

CONSERVATION RELATIONS FOR LOCAL THETA CORRESPONDENCE

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To the memory of Stephen Rallis

ABSTRACT. We prove Kudla-Rallis conjecture on first occurrences of local theta correspondence, for all type I irreducible dual pairs and all local fields of characteristic zero.

1. INTRODUCTION AND MAIN RESULTS

Fix a triple $(D \supset F, \epsilon)$, where F is a local field of characteristic 0; D is either F , or a quadratic field extension of F , or a central division quaternion algebra over F ; and $\epsilon = \pm 1$. Denote by ι the involutive anti-automorphism of D which is respectively the identity map, the non-trivial Galois element, or the main involution.

Throughout this article, let V be an ϵ -Hermitian right D -vector space, namely, V is a finite dimensional right D -vector space, equipped with a non-degenerate F -bilinear map

$$\langle, \rangle_V : V \times V \rightarrow D$$

satisfying

$$\langle u, va \rangle_V = \langle u, v \rangle_V a, \quad u, v \in V, a \in D,$$

and

$$\langle u, v \rangle_V = \epsilon \langle v, u \rangle_V^\iota, \quad u, v \in V.$$

Denote by $U(V)$ the isometry group of V , namely the group of D -linear automorphisms of V preserving the form \langle, \rangle_V . It is a classical group, as summarized in the following table:

D	F	quadratic extension	quaternion algebra
$\epsilon = 1$	orthogonal group	unitary group	quaternionic symplectic group
$\epsilon = -1$	symplectic group	unitary group	quaternionic orthogonal group

For convenience, we refer to the various cases by the types of the isometry groups in question.

2000 *Mathematics Subject Classification.* 22E46, 22E50 (Primary).

Key words and phrases. local theta correspondence, first occurrence, conservation relation, oscillator representation, degenerate principal series.

This work began in March, 2012 at the Institute for Mathematical Sciences (IMS), National University of Singapore. The authors thank IMS for the support.

Put

$$U'(V) := \begin{cases} \widetilde{\mathrm{Sp}}(V), & \text{if } U(V) \text{ is a nontrivial symplectic group and } F \not\cong \mathbb{C}; \\ U(V), & \text{otherwise.} \end{cases}$$

Here $\widetilde{\mathrm{Sp}}(V)$ denotes the metaplectic group, namely, the unique non-split topological central extension of the symplectic group $\mathrm{Sp}(V)$ by $\{\pm 1\}$.

By an enhanced oscillator representation of $U'(V)$, we mean a pair (W, ω) , where W is an ϵ -skew-Hermitian left D -vector space, and ω is a smooth oscillator representation of the Jacobi group

$$J'_V(W) := U'(V) \ltimes ((V \otimes_D W) \times F),$$

associated to a fixed nontrivial unitary character $\psi : F \rightarrow \mathbb{C}^\times$. See Section 2.1 for details. Similarly denote by $U(W)$ the isometry group of W , then $(U(V), U(W))$ is a type I irreducible dual pair in the sense of Howe [Ho1].

Two enhanced oscillator representations (W_1, ω_1) and (W_2, ω_2) of $U'(V)$ are said to be isomorphic if there is an isometric isomorphism $W_1 \cong W_2$ such that ω_1 is isomorphic to ω_2 with respect to the induced isomorphism $J'_V(W_1) \cong J'_V(W_2)$. Denote by Ω_V the set of isomorphism classes of enhanced oscillator representations of $U'(V)$. By abuse of notation, we do not distinguish an enhanced oscillator representation with its isomorphism class in Ω_V . We also do not distinguish a representation with its underlying vector space. The set Ω_V has a natural additive structure which makes it a commutative semi-group (see Section 2.1):

$$(W_1, \omega_1) + (W_2, \omega_2) := (W_1 \oplus W_2, \omega_1 \widehat{\otimes} \omega_2).$$

Here and henceforth, “ $\widehat{\otimes}$ ” stands for the completed projective tensor product in the archimedean case, and the algebraic tensor product in the non-archimedean case.

For $\sigma = (W, \omega) \in \Omega_V$, we shall refer to the dimension and the split rank of W as the dimension and the split rank of σ :

$$\dim \sigma := \dim_D W,$$

$$\mathrm{rank} \sigma := \mathrm{rank} W := \max \{ \dim_D Y \mid Y \text{ is a totally isotropic } D\text{-subspace of } W \}.$$

We say that σ splits if

- W splits, that is, $\dim \sigma = 2 \mathrm{rank} \sigma$, and
- for some (and hence all) Lagrangian subspaces Y of W , there is a nonzero (continuous in the archimedean case) linear functional on ω which is invariant under $U'(V) \ltimes (V \otimes_D Y) \subset J'_V(W)$.

As usual, “Lagrangian” means that Y is totally isotropic of dimension $\frac{\dim_D W}{2}$. There is exactly one isomorphism class of split enhanced oscillator representations in each even dimension, and the dimension map

$$\dim : \{ \sigma \in \Omega_V \mid \sigma \text{ splits} \} \xrightarrow{\sim} \{0, 2, 4, 6, \dots\}$$

is a semigroup isomorphism.

We say that two elements of the commutative semi-group Ω_V belong to the same Witt tower if they differ by a split element, that is, one of them is obtained from the other by adding a split element of Ω_V . This defines an equivalence relation in Ω_V whose equivalence classes are called Witt towers. The set of Witt towers in Ω_V , which is denoted by \mathfrak{W}_V , is a quotient group of the semi-group Ω_V . For $V = 0$, the group \mathfrak{W}_0 is the usual Witt group of ϵ -skew-Hermitian left D-vector spaces. In general, it is an extension of \mathfrak{W}_0 by the character group of $U(V)$:

$$1 \rightarrow \text{Hom}(U(V), \mathbb{C}^\times) \rightarrow \mathfrak{W}_V \rightarrow \mathfrak{W}_0 \rightarrow 1.$$

We refer the reader to Section 2.2 for details.

As a subset of Ω_V , each Witt tower \mathfrak{t} has the following convenient description. There is a unique $\sigma \in \mathfrak{t}$ with the minimum dimension, to be called the anisotropic degree of \mathfrak{t} :

$$\deg \mathfrak{t} := \min \{ \dim \sigma \mid \sigma \in \mathfrak{t} \}.$$

An enhanced oscillator representation $\sigma = (W, \omega) \in \mathfrak{t}$ is of dimension $\deg \mathfrak{t}$, if and only if W is anisotropic. The dimension map then yields a bijection:

$$\dim : \{ \sigma \mid \sigma \in \mathfrak{t} \} \xrightarrow{\sim} \{ \deg \mathfrak{t}, \deg \mathfrak{t} + 2, \deg \mathfrak{t} + 4, \dots \}.$$

Denote by $\text{Irr}(U'(V))$ the set of isomorphism classes of irreducible admissible smooth representations of $U'(V)$ if F is non-archimedean, and the set of isomorphism classes of irreducible Casselman-Wallach representations of $U'(V)$ if F is archimedean. The reader may consult [Ca] and [Wal, Chapter 11] for more information about Casselman-Wallach representations.

We are interested in occurrences of irreducible representations in enhanced oscillator representations: for a given $\sigma = (W, \omega) \in \Omega_V$ and $\pi \in \text{Irr}(U'(V))$, one seeks to determine whether the following space is nonzero:

$$(1) \quad \Theta_\sigma(\pi) := \text{Hom}_{U'(V)}(\omega, \pi).$$

Here “Hom” stands for the space of (continuous in the archimedean case) linear intertwining maps.

If $U'(V)$ is a metaplectic group, the space $\Theta_\sigma(\pi)$ is nonzero only when the non-trivial element of the covering map $U'(V) \rightarrow U(V)$ acts through the scalar multiplication by $(-1)^{\dim \sigma}$ in π . When this happens, we say that π is genuine with respect to σ . By convention, if $U'(V)$ is not a metaplectic group, we say that every irreducible representation in $\text{Irr}(U'(V))$ is genuine with respect to every element in Ω_V . We say that π is genuine with respect to a Witt tower $\mathfrak{t} \in \mathfrak{W}_V$ if it is genuine with respect to some (and hence all) elements of \mathfrak{t} .

Assume that $\pi \in \text{Irr}(U'(V))$ is genuine with respect to $\mathfrak{t} \in \mathfrak{W}_V$. There are two basic properties concerning non-vanishing of theta liftings:

- Non-vanishing of theta liftings in the stable range:

$$\text{if } \sigma \in \mathfrak{t} \text{ and } \text{rank } \sigma \geq \dim_{\mathbb{D}} V, \text{ then } \Theta_\sigma(\pi) \neq 0.$$

- Kudla's persistence principle:

if $\sigma_1, \sigma_2 \in \mathfrak{t}$ and $\dim \sigma_2 \geq \dim \sigma_1$, then $\Theta_{\sigma_1}(\pi) \neq 0$ implies $\Theta_{\sigma_2}(\pi) \neq 0$.

See Section 6.1 for details. Define the first occurrence index

$$(2) \quad n_{\mathfrak{t}}(\pi) := \min\{\dim \sigma \mid \sigma \in \mathfrak{t}, \Theta_{\sigma}(\pi) \neq 0\}.$$

In view of the aforementioned two properties, the first occurrence index is finite and is of clear interest.

Now we introduce some quantities which appear in the conservation relations. For $r \in \mathbb{Z}_{\geq 0}$, put

$$(3) \quad \rho_r := \rho_{r,D,F,\epsilon} := \begin{cases} \frac{2r-2}{4}, & \text{orthogonal group case;} \\ \frac{2r-1}{4}, & \text{quaternionic orthogonal group case;} \\ \frac{2r}{4}, & \text{unitary group case;} \\ \frac{2r+1}{4}, & \text{quaternion symplectic group case;} \\ \frac{2r+2}{4}, & \text{symplectic group case.} \end{cases}$$

When $r > 0$, $2\rho_r$ is the normalized exponent of the modulus character of a Siegel parabolic subgroup of $U(V_{r,r})$, where $V_{r,r}$ is the split ϵ -Hermitian right D-vector space of rank r . See Section 4.1. We note the numbers $4\rho_1 = 0, 1, 2, 3, 4$, which will appear in the sequel frequently. We also note that when F is non-archimedean, $4\rho_1$ coincides with the maximal dimension of anisotropic ϵ -skew-Hermitian left D-vector spaces, and up to isometric isomorphism, there is a unique such space of dimension $4\rho_1$.

In Section 3, we introduce the central object of this article: a subgroup $\mathcal{K}_V \subset \mathfrak{W}_V$, which we call the Kudla kernel in \mathfrak{W}_V .

First assume that F is non-archimedean. Then \mathcal{K}_V has order 2, unless $U(V)$ is the trivial orthogonal group (in which case $\mathcal{K}_V = \mathfrak{W}_V = 0$). Denote by \mathfrak{t}_V the generator of \mathcal{K}_V , and call it the anti-split Witt tower in \mathfrak{W}_V . It may be characterized as the unique element of \mathfrak{W}_V such that

$$(4) \quad \deg \mathfrak{t}_V = 4\rho_1 \quad \text{and} \quad n_{\mathfrak{t}_V}(1_V) = 4\rho_{\dim_D V} + 2.$$

Here and throughout this article, $1_V \in \text{Irr}(U'(V))$ stands for the trivial representation. The first equality in (4) says that \mathfrak{t}_V has maximal anisotropic degree. In view of the obvious equality

$$4\rho_{\dim_D V} + 2 = 4\rho_1 + 2\dim_D V,$$

the second equality in (4) asserts that the trivial representation only occurs in \mathfrak{t}_V when it has to, namely in the stable range. The element \mathfrak{t}_V is explicitly determined in all cases. For example, when $U(V)$ is an orthogonal group, $\mathfrak{t}_V \in \mathfrak{W}_V = \text{Hom}(O(V), \mathbb{C}^\times)$ is the sign character, and when $U(V)$ is a symplectic group, $\mathfrak{t}_V \in \mathfrak{W}_V = \mathfrak{W}_0$ is represented by the division quaternion algebra over F .

In the non-archimedean case, the conservation relations, as conjectured by Kudla and Rallis, assert the following:

Theorem A. *Assume that F is non-archimedean. Let \mathfrak{t}_1 and \mathfrak{t}_2 be two Witt towers in \mathfrak{W}_V with difference \mathfrak{t}_V . Then for every $\pi \in \text{Irr}(U'(V))$ which is genuine with respect to \mathfrak{t}_1 (and hence genuine with respect to \mathfrak{t}_2), one has that*

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) = 4\rho_{\dim_{\mathbb{D}} V} + 2.$$

Now we consider the archimedean case. Then three different phenomena occur.

Case I: $U(V)$ is a real or complex orthogonal group. Then as in the non-archimedean orthogonal case, $\mathcal{K}_V = \{1, \mathfrak{t}_V\}$ and \mathfrak{t}_V corresponds to the sign character of $O(V)$. The same conservation relations hold:

Theorem B. *Assume that $U(V)$ is a real or complex orthogonal group. Let \mathfrak{t}_1 and \mathfrak{t}_2 be two Witt towers in \mathfrak{W}_V with difference \mathfrak{t}_V . Then for every $\pi \in \text{Irr}(O(V))$ one has that*

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) = 4\rho_{\dim_{\mathbb{D}} V} + 2 \quad (= 2 \dim_{\mathbb{D}} V).$$

Case II: $U(V)$ is a complex symplectic group or a real quaternionic symplectic group. Then $\text{Hom}(U(V), \mathbb{C}^\times) = \{1\}$, and

$$\mathcal{K}_V = \mathfrak{W}_V = \mathfrak{W}_0 = \{0 = \mathfrak{t}_V^{\text{even}}, \mathfrak{t}_V = \mathfrak{t}_V^{\text{odd}}\}$$

is a group of order 2, where $\mathfrak{t}_V^{\text{even}}$ and $\mathfrak{t}_V^{\text{odd}}$ are the Witt towers of even and odd dimensional spaces, respectively. We may check from the explicit duality correspondence ([AB1, Section 2] and [LPTZ, Section 5]) that conservation relations fail in this case. For example, if V is a complex symplectic space of dimension 2, then $n_{\mathfrak{t}_V^{\text{even}}}(\pi) + n_{\mathfrak{t}_V^{\text{odd}}}(\pi)$ equals 3 if π is the trivial representation or one of the two irreducible constituents of the oscillator representation of $U(V) \cong \text{SL}_2(\mathbb{C})$, and equals 5 otherwise.

Case III: $U(V)$ is a real symplectic group, a real unitary group, or a real quaternionic orthogonal group. Then $\mathcal{K}_V \cong \mathbb{Z}$, and we have the following

Theorem C. *Assume that $U(V)$ is a real symplectic group, a real unitary group, or a real quaternionic orthogonal group. Let $\mathcal{T} \subset \mathfrak{W}_V$ be a \mathcal{K}_V -coset. Let $\pi \in \text{Irr}(U'(V))$ be an irreducible representation which is genuine with respect to some (and hence all) elements of \mathcal{T} . Then there are two different elements $\mathfrak{t}, \mathfrak{t}' \in \mathcal{T}$ such that*

$$n_{\mathfrak{t}}(\pi) + n_{\mathfrak{t}'}(\pi) = 4\rho_{\dim_{\mathbb{D}} V} + 2;$$

and for any two different elements $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}$, one has that

$$n_{\mathfrak{t}_1}(\pi) + n_{\mathfrak{t}_2}(\pi) \geq 4\rho_{\dim_{\mathbb{D}} V} + 2 + 4\rho_1(|\mathfrak{t}_1 - \mathfrak{t}_2| - 1),$$

where for $\mathbf{t} \in \mathcal{K}_V$, $|\mathbf{t}|$ denotes the nonnegative integer so that \mathbf{t} is $|\mathbf{t}|$ -multiple of a generator of \mathcal{K}_V . Consequently the following conservation relations hold:

$$\sum_{\mathcal{Q} \in \mathcal{T}/2\mathcal{K}_V} \min\{n_{\mathbf{t}}(\pi) \mid \mathbf{t} \in \mathcal{Q}\} = 4\rho_{\dim_{\mathbb{D}} V} + 2.$$

Remarks: (a) For orthogonal-symplectic and unitary-unitary dual pairs, the conservation relations were conjectured by Kudla and Rallis in the mid 1990's. For quaternionic dual pairs, the conjectured statements first appeared in Gan-Tantono [GTan, Section 4].

(b) For p -adic orthogonal-symplectic dual pairs and π supercuspidal, the conservation relations were due to Kudla and Rallis [KR3]. This was later extended to all type I irreducible dual pairs by Minguez [Mi], again for p -adic F and π supercuspidal.

(c) The inequality $n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) \geq 4\rho_{\dim_{\mathbb{D}} V} + 2$ is due to Kudla-Rallis [KR3, Theorem 3.8] (for p -adic orthogonal-symplectic dual pairs) and Gong-Grenié [GG, Theorem 1.8] (for p -adic unitary-unitary dual pairs, following an earlier work of Harris-Kudla-Sweet [HKS]). See also Gan-Ichino [GI, Theorem 5.4].

(d) The conservation relations for complex orthogonal groups were proved by Adams-Barbasch [AB1] using the explicit duality correspondence. Also for $F = \mathbb{R}$, A. Paul proved the conservation relations for unitary-unitary dual pair correspondence [Pa2, Theorem 1.4], for a discrete series representation, or a representation irreducibly induced from a discrete series representation.

In addition to the above historical remarks, the authors would like to emphasize the deep influence of the ideas of Kudla and Rallis on this article, in particular the use of degenerate principal series. The proof of our results follows their approach [KR3]. There are two ingredients: an upper bound and a lower bound. For the upper bound, our main contribution (Proposition 4.2) is to pinpoint and to recognize the role of certain structure results about degenerate principal series representations, which fortunately can be read off from results in the existing literature. It identifies the (individual) contribution of a certain submodule R_{σ} (first employed in the influential paper of Rallis [Ra1]) in the degenerate principal series. This will imply the required existence/occurrence result (Proposition 6.5). Note that Proposition 4.2 does not hold when $U(V)$ is a complex symplectic group or a real quaternionic symplectic group. Our second main contribution is to introduce the notion of Kudla kernel. The key technical result (Proposition 5.1), as remarked previously, is that for the Witt tower represented by a nontrivial element of the Kudla kernel, the trivial representation only occurs in the stable range. This is responsible for the lower bound. For non-archimedean orthogonal-symplectic and unitary-unitary dual pairs, Proposition 5.1 is a reformulation of results of Rallis, Kudla-Rallis and Gong-Grenié. All archimedean cases are also

known [Pr2, Pa2, LPTZ, LL]. So only the non-archimedean quaternionic case of Proposition 5.1 is really new. Note that due to the lack of MVW-involutions, the approach of Kudla-Rallis and Gong-Grenié, which uses the doubling method, does not work for this case. We follow the idea of Rallis ([Ra1, Ra2], which treat the case of orthogonal groups) to provide a uniform proof of Proposition 5.1 in all non-archimedean cases. We remark that our methods to establish both upper and lower bounds are invariant-theoretic: ultimately the upper bound rests on the existence, and the lower bound rests on the non-existence, of certain invariant distributions.

As pointed out by Kudla and Rallis [KR3], the conservation relations imply theta dichotomy phenomenon in the non-archimedean cases. The latter was established by Harris [Ha, Theorem 2.1.7] (for unitary-unitary dual pairs), and Zorn [Zo, Theorem 1.1], and by Gan-Gross-Prasad [GGP, Theorem 11.1] (for orthogonal-symplectic dual pairs). For a related work of Prasad, see [Pra]. For $F = \mathbb{R}$, the corresponding (though more complicated) result was established by Adams-Babasch [AB2, Corollary 5.3], Paul [Pa1, Introduction], and Li-Paul-Tan-Zhu [LPTZ, Introduction], using Vogan's version of the Langlands classification. See also the work of Mœglin [Mo1]. The conservation relations proved in this article thus yield a shorter proof for the afore-mentioned (archimedean) results, as one of the by-products. Other works where conservation relations are exploited include those of Gan-Takeda [GTak], Mœglin [Mo2], and Gan-Qiu-Takeda [GQT].

The article is organized as follows. In Section 2, we introduce the semigroup of enhanced oscillator representations, the associated Witt group, and we discuss the natural operation of restrictions. In Section 3, we introduce the notion of Kudla characters and Kudla kernels, and we determine the latter explicitly in all cases. In Section 4, we review some structure results of degenerate principal series of $U'(V)$ for V split and the doubling method, along with some technical preparations towards non-vanishing of theta liftings. Section 5 is devoted to the phenomenon of non-occurrence of the trivial representation before stable range. As pointed out earlier, the idea is due to Rallis and it consists of reduction to the null cone, and within the null cone, proving vanishing on small orbits and homogeneity for the main orbits, and finally employing the Fourier transform. It is worth mentioning that for the base case ($\dim_{\mathbb{D}} V = 1$, and W anisotropic), proving non-occurrence of the trivial representation (Lemma 5.3) again requires the use of the doubling method. In the final Section 6, we derive the conservation relations by proving an upper bound and a lower bound, as consequences of the results of Sections 4 and 5 respectively.

Finally the authors would like to thank Wee Teck Gan, Michael Harris, Atsushi Ichino and Jian-Shu Li for their interest, as well as comments on an earlier version of this article. The authors owe a special thanks to Dipendra Prasad for sending us the preprint of Rallis [Ra2], in addition to his interest and comments.

2. ENHANCED OSCILLATOR REPRESENTATIONS

2.1. The semigroup of enhanced oscillator representations. Let W be an ϵ -skew-Hermitian left D -vector space, namely, W is a finite dimensional left D -vector space, equipped with a non-degenerate F -bilinear map

$$\langle \cdot, \cdot \rangle_W : W \times W \rightarrow D$$

satisfying

$$\langle au, v \rangle_W = a \langle u, v \rangle_W, \quad u, v \in W, a \in D,$$

and

$$\langle u, v \rangle_W = -\epsilon \langle v, u \rangle_W^t, \quad u, v \in W.$$

Recall from the Introduction that V is an ϵ -Hermitian right D -vector space. The tensor product $V \otimes_D W$ is a symplectic space over F under the form

$$\langle v \otimes w, v' \otimes w' \rangle_{V \otimes_D W} := \frac{\langle v, v' \rangle_V \langle w, w' \rangle_W^t + \langle w, w' \rangle_W \langle v, v' \rangle_V^t}{2}, \quad v, v' \in V, w, w' \in W.$$

Define the Heisenberg group

$$(5) \quad H_V(W) := (V \otimes_D W) \times F,$$

whose group multiplication is given by

$$(u, t)(u', t') := (u + u', t + t' + \langle u, u' \rangle_{V \otimes_D W}), \quad u, u' \in V \otimes_D W, t, t' \in F.$$

Form the semidirect product

$$(6) \quad J_V(W) := U(V) \ltimes H_V(W)$$

and its covering

$$(7) \quad J'_V(W) := U'(V) \ltimes H_V(W).$$

Fix a non-trivial unitary character $\psi : F \rightarrow \mathbb{C}^\times$ throughout the article. A representation of $J'_V(W)$ is called a smooth oscillator representation (associated to ψ , unless otherwise specified) if

- it is a smooth representation if F is non-archimedean, and a smooth Fréchet representation of moderate growth if F is archimedean;
- as a representation of $H_V(W)$, it is irreducible with central character ψ .

The reader is referred to [du, Definition 1.4.1] or [Su2, Section 2] for the notion of “smooth Fréchet representations of moderate growth” in the setting of Jacobi groups. Smooth oscillator representations of $J'_V(W)$ exist by the well known result of splitting metaplectic covers ([MVW], see also [Ku1, Proposition 4.1]). Denote by $\Omega_V(W)$ the set of isomorphism classes of smooth oscillator representations of $J'_V(W)$. We shall use the obvious notion of smooth oscillator representations for some other semidirect products of reductive groups with Heisenberg groups.

Similarly, we define the groups $U(W)$, $U'(W)$ and

$$J'_{V,W} := (U'(V) \times U'(W)) \ltimes H_V(W) \supset J'_V(W).$$

Lemma 2.1. *Every smooth oscillator representation of $J'_V(W)$ extends to a smooth oscillator representation of $J'_{V,W}$.*

Proof. Besides splitting of metaplectic covers, this is due to the fact that any two elements in a metaplectic group commute with each other if their projections to the symplectic group commute with each other [MVW, Chapter 2, Lemma II.5]. \square

As in the Introduction, denote by Ω_V the set of isomorphism classes of enhanced oscillator representations of $U'(V)$.

Lemma 2.2. *Two smooth oscillator representations ω_1 and ω_2 of $J'_V(W)$ are isomorphic to each other if and only if (W, ω_1) and (W, ω_2) are isomorphic to each other as enhanced oscillator representations of $U'(V)$.*

Proof. The “only if” part is trivial. To prove the “if” part, assume that (W, ω_1) and (W, ω_2) are isomorphic to each other as enhanced oscillator representations. This amounts to saying that there is an element $g \in U(W)$ and a (continuous in the archimedean case) linear isomorphism $\phi : \omega_1 \rightarrow \omega_2$ such that the diagram

$$\begin{array}{ccc} \omega_1 & \xrightarrow{\phi} & \omega_2 \\ h \downarrow & & \downarrow g_V(h) \\ \omega_1 & \xrightarrow{\phi} & \omega_2 \end{array}$$

commutes for every $h \in J'_V(W)$, where $g_V : J'_V(W) \rightarrow J'_V(W)$ is the automorphism induced by $g : W \rightarrow W$.

Using Lemma 2.1, we extend ω_2 to a representation of $J'_{V,W}$, which we still denote by ω_2 . Let g' be an element of $U'(W)$ which lifts g . Then $g_V(h) = g'hg'^{-1}$ for every $h \in J'_V(W)$. Therefore the diagram

$$\begin{array}{ccc} \omega_1 & \xrightarrow{g'^{-1} \circ \phi} & \omega_2 \\ h \downarrow & & \downarrow h \\ \omega_1 & \xrightarrow{g'^{-1} \circ \phi} & \omega_2 \end{array}$$

commutes for every $h \in J'_V(W)$, and consequently, ω_1 and ω_2 are isomorphic as representations of $J'_V(W)$. \square

Lemma 2.2 implies that $\Omega_V(W)$ only depends on the isometric class of W , that is, all isometric isomorphisms $W_1 \cong W_2$ induce the same bijection $\Omega_V(W_1) \cong \Omega_V(W_2)$. It also implies that

$$\Omega_V = \bigsqcup_W \Omega_V(W),$$

where W runs through all isometric classes of ϵ -skew-Hermitian left D-vector spaces.

For any two elements $\sigma_1 = (W_1, \omega_1)$ and $\sigma_2 = (W_2, \omega_2)$ in Ω_V , put

$$\sigma_1 + \sigma_2 := (W_1 \oplus W_2, \omega_1 \widehat{\otimes} \omega_2) \in \Omega_V.$$

Note that $\omega_1 \widehat{\otimes} \omega_2$ is a smooth oscillator representation of $J'_V(W_1 \oplus W_2)$: it is a tensor product representation of $U'(V)$, and is a representation of $H_V(W_1 \oplus W_2)$ by descending the representation of $H_V(W_1) \times H_V(W_2)$ through the quotient map

$$H_V(W_1) \times H_V(W_2) \rightarrow H_V(W_1 \oplus W_2), \quad (u_1, t_1; u_2, t_2) \mapsto (u_1 + u_2, t_1 + t_2).$$

The operation “+” defines a commutative semigroup structure on Ω_V .

We may also define the contragredient operation on Ω_V by

$$(8) \quad (W, \omega)^\vee := (W^-, \omega^\vee) \in \Omega_V,$$

where W^- is the space W equipped with the form scaled by -1 , and ω^\vee is the smooth oscillator representation of $J'_V(W^-)$ which is contragredient to ω with respect to the isomorphism

$$\begin{aligned} J'_V(W^-) = U'(V) \ltimes ((V \otimes_{\mathbb{D}} W^-) \times F) &\rightarrow J'_V(W) = U'(V) \ltimes ((V \otimes_{\mathbb{D}} W) \times F), \\ (g; u, t) &\mapsto (g; u, -t). \end{aligned}$$

When $W = 0$, a smooth oscillator representation of $J'_V(0)$ is just a character of $U(V)$:

$$(9) \quad \Omega_V(0) = \text{Hom}(U'(V), \mathbb{C}^\times) = \text{Hom}(U(V), \mathbb{C}^\times).$$

Note that $\Omega_V(0)$ is a subgroup of the semigroup Ω_V . We remark that all characters of $U(V)$ are unitary, and all smooth oscillator representations of $J'_V(W)$ are unitarizable.

2.2. The Witt group of enhanced oscillator representations. We use the following notation throughout this article: given $(W, \omega) \in \Omega_V$ and a Lagrangian subspace \mathbf{L} of the symplectic space $V \otimes_{\mathbb{D}} W$, let $\lambda_{\mathbf{L}}$ denote a nonzero (continuous in the archimedean case) linear functional on ω which is invariant under $\mathbf{L} \subset H_V(W)$. It is unique up to multiples. See [Ho2, Theorem 5.1].

Let $\sigma = (W, \omega) \in \Omega_V$. Suppose that W is split. We pick a Lagrangian subspace Y of W , and define a character χ_σ on $U'(V)$ by the formula

$$(10) \quad \lambda_{V \otimes_{\mathbb{D}} Y}(g \cdot v) = \chi_\sigma(g) \lambda_{V \otimes_{\mathbb{D}} Y}(v), \quad g \in U'(V), v \in \omega.$$

Lemma 2.1 implies that the character χ_σ is independent of Y . We therefore have a semigroup homomorphism:

$$(11) \quad \bigsqcup_{W \text{ splits}} \Omega_V(W) \rightarrow \text{Hom}(U'(V), \mathbb{C}^\times), \quad \sigma \mapsto \chi_\sigma.$$

Note that the restriction of (11) to $\Omega_V(0)$ coincides with the identification (9).

Lemma 2.3. *For every ϵ -skew-Hermitian left D-vector space W , the action*

$$\Omega_V(0) \times \Omega_V(W) \rightarrow \Omega_V(W), \quad (\chi, \sigma) \mapsto \chi + \sigma$$

is simply transitive.

Proof. Transitivity is clear. Assume that $\chi + \sigma = \sigma$. Then

$$\chi + (\sigma + \sigma^\vee) = \sigma + \sigma^\vee.$$

We conclude that $\chi = 1$ by applying the homomorphism (11) to the above. \square

As in the Introduction, $\sigma = (W, \omega) \in \Omega_V$ is said to be split if W is split and χ_σ is trivial. It is clear that $\sigma + \sigma^\vee$ splits for all $\sigma \in \Omega_V$.

Lemma 2.4. *Let $\sigma_1, \sigma_2 \in \Omega_V$. If both σ_2 and $\sigma_1 + \sigma_2$ split, then so does σ_1 .*

Proof. Write $\sigma_1 = (W_1, \omega_1)$. Witt's cancellation theorem implies that W_1 splits, and the equalities

$$1 = \chi_{\sigma_2} = \chi_{\sigma_1 + \sigma_2} = \chi_{\sigma_1} \chi_{\sigma_2}$$

implies that $\chi_{\sigma_1} = 1$. \square

Recall also that two elements $\sigma_1, \sigma_2 \in \Omega_V$ are said to be in the same Witt tower if they differ by a split element. By Lemma 2.4, this amounts to saying that $\sigma_1 + \sigma_2^\vee$ splits, and defines an equivalence relation on Ω_V . An equivalent class of this relation is called a Witt tower in Ω_V .

Denote by \mathfrak{W}_V the set of Witt towers in Ω_V . This is a quotient group of the semigroup Ω_V . Its zero element is the Witt tower of split elements, and the inverse map is induced by the contragredient operation $\sigma \mapsto \sigma^\vee$.

Note that when $V = 0$, \mathfrak{W}_0 equals the Witt group of ϵ -skew-Hermitian left D-vector spaces, which is well understood in all cases (cf. [Sch]). The following sequence of abelian groups is exact:

$$(12) \quad 1 \rightarrow \Omega_V(0) \rightarrow \mathfrak{W}_V \xrightarrow{q_V} \mathfrak{W}_0 \rightarrow 1,$$

where $q_V : \mathfrak{W}_V \rightarrow \mathfrak{W}_0$ is the map of sending the class of (W, ω) to the Witt class of W . Note also that the character group $\Omega_V(0)$ is explicitly calculated in all cases (cf. [Wa]).

2.3. Restrictions of enhanced oscillator representations. Let V' be an ϵ -Hermitian right D-vector space so that V is isometrically embedded in V' :

$$V \hookrightarrow V'.$$

Denote by V^\perp the orthogonal complement of V in V' . Then the induced embedding $U(V) \times U(V^\perp) \hookrightarrow U(V')$ uniquely lifts to a homomorphism

$$(13) \quad U'(V) \times U'(V^\perp) \rightarrow U'(V').$$

For every ϵ -skew-Hermitian left D-vector space W , combining (13) with the homomorphism

$$H_V(W) \times H_{V^\perp}(W) \rightarrow H_{V'}(W), \quad (u, t; u', t') \mapsto (u + u', t + t'),$$

we obtain a homomorphism

$$(14) \quad J'_V(W) \times J'_{V^\perp}(W) \rightarrow J'_{V'}(W).$$

The restriction of every $\omega' \in \Omega_{V'}(W)$ through (14) is uniquely of the form

$$(15) \quad \omega'|_{J'_V(W) \times J'_{V^\perp}(W)} = \omega'|_V \widehat{\otimes} \omega'|_{V^\perp},$$

where $\omega'|_V \in \Omega_V(W)$ and $\omega'|_{V^\perp} \in \Omega_{V^\perp}(W)$ are defined via the above. In turn this defines a semigroup homomorphism

$$(16) \quad r_V^{V'} : \Omega_{V'} \rightarrow \Omega_V, \quad (W, \omega') \mapsto (W, \omega'|_V).$$

It further descends to a group homomorphism

$$(17) \quad r_V^{V'} : \mathfrak{W}_{V'} \rightarrow \mathfrak{W}_V.$$

It is clear that the diagrams in

$$(18) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \Omega_{V'}(0) & \longrightarrow & \mathfrak{W}_{V'} & \longrightarrow & \mathfrak{W}_0 \longrightarrow 1 \\ & & \downarrow r_V^{V'} & & \downarrow r_V^{V'} & & \parallel \\ 1 & \longrightarrow & \Omega_V(0) & \longrightarrow & \mathfrak{W}_V & \longrightarrow & \mathfrak{W}_0 \longrightarrow 1 \end{array}$$

are commutative, where the map $r_V^{V'}$ of left vertical arrow is the restriction of characters. Therefore the group \mathfrak{W}_V is determined by $\mathfrak{W}_{V'}$:

$$\mathfrak{W}_V = \frac{\Omega_V(0) \oplus \mathfrak{W}_{V'}}{\{(r_V^{V'}(\chi), \chi^{-1}) \mid \chi \in \Omega_{V'}(0)\}}.$$

Lemma 2.5. *The restriction maps in (16) and (17) are independent of the isometric embedding $V \hookrightarrow V'$. Furthermore the restriction maps are isomorphisms in the following cases:*

- (i) V is a symplectic space or a quaternionic Hermitian space;
- (ii) V is a nonzero Hermitian or skew-Hermitian space;
- (iii) V is a non-anisotropic orthogonal space or a non-anisotropic quaternionic skew-Hermitian space.

Proof. If $V' = V$, then every isometric isomorphism $V \rightarrow V$ induces an inner automorphism $J'_V(W) \rightarrow J'_V(W)$, and consequently, both (16) and (17) are identity maps. In general, use Witt's extension theorem and apply the above argument to V' , we conclude that the maps in (16) and (17) are independent of the embedding $V \hookrightarrow V'$. This proves the first assertion.

The second assertion follows from the commutativity of the diagrams in (18), and the calculation of the character groups of classical groups (cf. [Wa, Section 2]). \square

By the above lemma, the semigroup Ω_V and the group \mathfrak{W}_V only depend on the isometric class of V , namely, if V' is isometrically isomorphic to V , then all different isometric isomorphisms induce the same semigroup isomorphisms $\Omega_V \cong \Omega_{V'}$, and the same group isomorphism $\mathfrak{W}_V \cong \mathfrak{W}_{V'}$.

If V' is non-anisotropic, we denote

$$(19) \quad \mathfrak{W}'_V := r_V^{V'}(\mathfrak{W}_{V'}) \subset \mathfrak{W}_V.$$

It is independent of V' by Lemma 2.5.

Lemma 2.6. *We have that*

$$\mathfrak{W}_V = \mathfrak{W}'_V + \Omega_V(0).$$

Proof. This is because that the map $q_V : \mathfrak{W}_V \rightarrow \mathfrak{W}_0$ has kernel $\Omega_V(0)$ and is surjective when restricted to \mathfrak{W}'_V . \square

3. KUDLA CHARACTERS AND KUDLA KERNELS

In Sections 3.1 to 3.3, we assume that V is split and nonzero. We define the notion of the Kudla character of $\mathfrak{t} \in \mathfrak{W}_V$ and concretely realize it as a character of a compact abelian group K' . In Section 3.4, we introduce the Kudla kernel in \mathfrak{W}_V for an arbitrary V . This is defined by way of restriction from a nonzero split V' using the Kudla character map of $\mathfrak{W}_{V'}$.

3.1. Kudla characters of enhanced oscillator representations. For every Lagrangian subspace X of V , denote by $P(X)$ the parabolic subgroup of $U(V)$ stabilizing X , and by $P'(X) \rightarrow P(X)$ the covering induced by $U'(V) \rightarrow U(V)$.

We use $|\cdot|_X$ to denote the following positive character on $P'(X)$:

$$(20) \quad P'(X) \rightarrow P(X) \xrightarrow{\text{restriction on } X} GL(X) \xrightarrow{\det} E^\times \xrightarrow{|\cdot|_E} \mathbb{R}_+^\times \xrightarrow{x \mapsto x^{d_D}} \mathbb{R}_+^\times.$$

Here and henceforth E is the center of D , “det” stands for the reduced norm, $|\cdot|_E$ is the normalized absolute value on E , and d_D is the degree of D over E , which is 2 if D is quaternion, and is 1 otherwise.

Let $\sigma = (W, \omega) \in \Omega_V$. Define a (unitary) character $\kappa_{\sigma, X}$ on $P'(X)$ by the formula

$$(21) \quad \lambda_{X \otimes_D W}(g \cdot v) = \kappa_{\sigma, X}(g) |g|_X^{\frac{\dim \sigma}{2}} \lambda_{X \otimes_D W}(v), \quad g \in P'(X), v \in \omega.$$

Lemma 3.1. *If Y is another Lagrangian subspace of V , then*

$$\kappa_{\sigma, X}(g) = \kappa_{\sigma, Y}(g)$$

for all $g \in P'(X) \cap P'(Y)$.

Proof. By Jordan decomposition, we may assume without loss of generality that g is semisimple, namely the image g_0 of g under the quotient map $U'(V) \rightarrow U(V)$ is semisimple. Then it is elementary to see that there is a g_0 -stable Lagrangian subspace Y' of V such that

$$V = X \oplus Y' \quad \text{and} \quad Y = (Y \cap X) \oplus (Y \cap Y').$$

We realize $\omega|_{H_V(W)}$ on the space of Schwartz half densities (cf. [Li, Section 1.1] for the notion of “Schwartz half densities”) on $X \otimes_D W$, with the following action:

- The action of $u \in X \otimes_{\mathbb{D}} W$ is the pushing forward of half densities through the translation by u .
- The action of $v \in Y' \otimes_{\mathbb{D}} W$ is multiplication by the function $\psi(\langle 2v, \cdot \rangle_{V \otimes W})$.

Denote by μ_X a Haar measure on $X \otimes_{\mathbb{D}} W$, and by $\mu_X^{1/2}$ its square root, which is a half density on $X \otimes_{\mathbb{D}} W$. Up to scalar multiplication, the functional $\lambda_{X \otimes_{\mathbb{D}} W}$ is given by

$$f \mu_X^{1/2} \mapsto \int_{X \otimes_{\mathbb{D}} W} f \mu_X,$$

where f is a Schwartz function on $X \otimes_{\mathbb{D}} W$. The functional $\lambda_{Y \otimes_{\mathbb{D}} W}$ is given by

$$f \mu_X^{1/2} \mapsto \int_{(X \cap Y) \otimes_{\mathbb{D}} W} f|_{(X \cap Y) \otimes_{\mathbb{D}} W} \mu_{X \cap Y},$$

where $\mu_{(X \cap Y)}$ is a Haar measure on $(X \cap Y) \otimes_{\mathbb{D}} W$.

Note that g acts as $c g_*$ under the oscillator representation ω , for some complex number c of modulus 1, where g_* is the pushing forward of half densities through the map $g_0 \otimes 1 : X \otimes_{\mathbb{D}} W \rightarrow X \otimes_{\mathbb{D}} W$. Now it is routine to verify that

$$\kappa_{\sigma, X}(g) = \kappa_{\sigma, Y}(g) = c.$$

□

By Lemma 3.1, we get a well defined map

$$(22) \quad \kappa_{\sigma} : U'(V)_{\text{split}} := \bigcup_{X \text{ is a Lagrangian subspace of } V} P'(X) \rightarrow \mathbb{C}^{\times},$$

which maps $g \in P'(X)$ to $\kappa_{\sigma, X}(g)$. Obviously the map κ_{σ} has the following two properties:

- it is $U'(V)$ -conjugation invariant;
- its restriction to each $P'(X)$ is a continuous group homomorphism.

For any abelian topological group A , denote by $\text{Hom}(U'(V)_{\text{split}}, A)$ the group of all maps from $U'(V)_{\text{split}}$ to A with the afore mentioned properties. We call

$$(23) \quad \kappa_{\sigma} \in \text{Hom}(U'(V)_{\text{split}}, \mathbb{C}^{\times})$$

the Kudla character of $\sigma \in \Omega_V$.

We omit the proof of the following lemma, which is similar to (but easier than) that of Lemma 3.1.

Lemma 3.2. *The map κ_{σ} is trivial if $\sigma \in \Omega_V$ splits. Consequently $\sigma \mapsto \kappa_{\sigma}$ descends to a group homomorphism*

$$(24) \quad \mathfrak{W}_V \rightarrow \text{Hom}(U'(V)_{\text{split}}, \mathbb{C}^{\times}), \quad \mathbf{t} \mapsto \kappa_{\mathbf{t}}.$$

3.2. **The group $\text{Hom}(U'(V)_{\text{split}}, \mathbb{C}^\times)$.** We first consider the case of linear groups. Put

$$U(V)_{\text{split}} := \bigcup_{X \text{ is a Lagrangian subspace of } V} P(X).$$

Similar to $\text{Hom}(U'(V)_{\text{split}}, A)$, we define the group $\text{Hom}(U(V)_{\text{split}}, A)$ for every abelian topological group A .

Put

$$(25) \quad K := \frac{\text{Image of the reduced norm map } D^\times \rightarrow E^\times}{\{xx^\iota \mid x \in E^\times\}}.$$

(Recall that E is the center of D .) It is always a compact abelian group. Define a map

$$\mu_V : U(V)_{\text{split}} \rightarrow K$$

by mapping $g \in P(X)$ to the class of $\det(g|_X)$ in K .

We omit the proof of the following elementary lemma.

Lemma 3.3. *The map μ_V is well-defined, belongs to $\text{Hom}(U(V)_{\text{split}}, K)$, and is surjective. Its pull-back yields a group isomorphism*

$$\text{Hom}(U(V)_{\text{split}}, A) \cong \text{Hom}(K, A)$$

for every abelian topological group A .

Now assume that we are in the case of metaplectic groups, that is, $D = F \not\cong \mathbb{C}$ and $\epsilon = -1$. We have a two-to-one map

$$(26) \quad \widetilde{\text{Sp}}(V)_{\text{split}} \rightarrow \text{Sp}(V)_{\text{split}},$$

and a surjective map

$$\mu_V : \text{Sp}(V)_{\text{split}} \rightarrow K = \frac{F^\times}{(F^\times)^2}.$$

Put

$$\widetilde{K} := K \times \{\pm 1\},$$

to be viewed as an abelian group with group multiplication

$$(a, t)(a', t') := (aa', tt'(a, a')_F), \quad \text{where } (\cdot, \cdot)_F \text{ is the Hilbert symbol for } F.$$

It is an extension of K by $\{\pm 1\}$, equipped with a family

$$(27) \quad \{\gamma_{\psi'}\}_{\psi'} \text{ is a non-trivial unitary character on } F$$

of characters, where

$$(28) \quad \gamma_{\psi'}(a, t) := t \frac{\gamma(x \mapsto \psi'(ax^2))}{\gamma(x \mapsto \psi'(x^2))},$$

and the two γ 's of the right hand side of (28) stand for Weil indices (see [Weil, Section 14] or [Weis]) of non-degenerate characters (on F) of degree two.

For every $\alpha \in F^\times$, denote by $\alpha\psi$ the character of F given by

$$(\alpha\psi)(x) = \psi(\alpha x), \quad x \in F.$$

Then it is known that [Rao, Corollary A.5.]

$$(29) \quad \gamma_\psi(a, t)\gamma_{\alpha\psi}(a, t) = (a, -\alpha)_F, \quad (a, t) \in \tilde{K}.$$

Recall that the Heisenberg group $H(V) := V \times F$ has the group multiplication

$$(u, t)(u', t') := (u + u', t + t' + \langle u, u' \rangle_V).$$

For every non-trivial unitary character ψ' of F , denote by $\omega_{\psi'}$ the smooth oscillator representation of $\widetilde{\mathrm{Sp}}(V) \ltimes H(V)$ associated to ψ' . As a special case of (21), we define a map $\kappa_{\psi'} \in \mathrm{Hom}(\widetilde{\mathrm{Sp}}(V)_{\mathrm{split}}, \mathbb{C}^\times)$ by

$$\lambda_{\psi', X}(g \cdot v) = \kappa_{\psi'}(g)|g|_X^{\frac{1}{2}}\lambda_{\psi', X}(v), \quad v \in \omega_{\psi'}, g \in P'(X),$$

for all Lagrangian subspaces X of V , where $\lambda_{\psi', X}$ denotes a nonzero (continuous in the archimedean case) linear functional on $\omega_{\psi'}$ which is invariant under $X \subset H(V)$.

Lemma 3.4. *There exists a unique map $\tilde{\mu}_V : \widetilde{\mathrm{Sp}}(V)_{\mathrm{split}} \rightarrow \tilde{K}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{\mathrm{Sp}}(V)_{\mathrm{split}} & \xrightarrow{\tilde{\mu}_V} & \tilde{K} \\ \downarrow \kappa_{\psi'} & & \downarrow \gamma_{\psi'} \\ \mathbb{C}^\times & \xlongequal{\quad} & \mathbb{C}^\times \end{array}$$

commutes for every non-trivial unitary character ψ' of F .

Proof. The uniqueness is clear since the family (27) of characters separates the elements of the group \tilde{K} . We prove the existence in what follows.

Let $g \in \widetilde{\mathrm{Sp}}(V)_{\mathrm{split}}$ and denote by g_0 its image under the map $\widetilde{\mathrm{Sp}}(V)_{\mathrm{split}} \rightarrow \mathrm{Sp}(V)_{\mathrm{split}}$. Let $\alpha \in F^\times$. By using the Schrodinger model of the representation $\omega_{\psi} \hat{\otimes} \omega_{\alpha\psi}$ of the group $\mathrm{Sp}(V)$ ([Ku2, Section II.4]), we have that

$$(30) \quad \kappa_\psi(g)\kappa_{\alpha\psi}(g) = (\mu_V(g_0), -\alpha)_F.$$

Then it is elementary to see that (29) and (30) imply that

$$\kappa_{\alpha\psi}(g) = \gamma_{\alpha\psi}(\mu_V(g_0), t_g),$$

where $t_g \in \{\pm 1\}$ is independent of α . Therefore the map $g \mapsto (\mu_V(g_0), t_g)$ fulfills the requirement of the lemma. \square

By Lemma 3.4 and its proof, the diagrams in

$$(31) \quad \begin{array}{ccccc} \{\pm 1\} & \longrightarrow & \widetilde{\mathrm{Sp}}(V)_{\mathrm{split}} & \longrightarrow & \mathrm{Sp}(V)_{\mathrm{split}} \\ & & \downarrow \tilde{\mu}_V & & \downarrow \mu_V \\ \{\pm 1\} & \longrightarrow & \tilde{K} & \longrightarrow & K \end{array}$$

are commutative. Therefore $\tilde{\mu}_V$ is surjective. It is easy to check that $\tilde{\mu}_V \in \text{Hom}(\widetilde{\text{Sp}}(V)_{\text{split}}, \tilde{K})$, and its pull-back yields a group isomorphism

$$\text{Hom}(\widetilde{\text{Sp}}(V)_{\text{split}}, A) \cong \text{Hom}(\tilde{K}, A)$$

for every abelian topological group A .

We are back in the general case. Put

$$K' := \begin{cases} \tilde{K}, & \text{if } D = F \not\cong \mathbb{C} \text{ and } \epsilon = -1; \\ K, & \text{otherwise,} \end{cases}$$

and put

$$\mu'_V := \begin{cases} \tilde{\mu}_V, & \text{if } D = F \not\cong \mathbb{C} \text{ and } \epsilon = -1; \\ \mu_V, & \text{otherwise.} \end{cases}$$

Then in all cases, descending through $\mu'_V : U'(V)_{\text{split}} \rightarrow K'$ yields an isomorphism

$$\text{Hom}(U'(V)_{\text{split}}, \mathbb{C}^\times) \xrightarrow{\sim} \hat{K}'.$$

Here and henceforth, “ $\hat{}$ ” over a compact abelian group indicates its character group. Finally, we get a homomorphism

$$(32) \quad \begin{aligned} \nu_V : \quad & \mathfrak{W}_V \rightarrow \hat{K}', \\ & \mathfrak{t} \mapsto \text{the descending of } \kappa_{\mathfrak{t}} \text{ through } \mu'_V. \end{aligned}$$

Lemma 3.5. *If V and V' are nonzero split ϵ -Hermitian right D -vector spaces so that $\dim_D V' \geq \dim_D V$. Then the restriction map*

$$r_V^{V'} : \mathfrak{W}_{V'} \rightarrow \mathfrak{W}_V$$

is an isomorphism and

$$\nu_{V'} = \nu_V \circ r_V^{V'}.$$

Proof. The first assertion follows from Lemma 2.5. Fix an isometric linear embedding $V \rightarrow V'$, then the second assertion is a consequence of the fact that the diagram

$$\begin{array}{ccc} U'(V)_{\text{split}} & \longrightarrow & U'(V')_{\text{split}} \\ \downarrow \mu'_V & & \downarrow \mu'_{V'} \\ K' & \xlongequal{\quad} & K' \end{array}$$

commutes. □

3.3. The homomorphism ν_V . In this subsection, we examine the homomorphism ν_V case by case. We have the usual dimension homomorphism

$$\dim : \mathfrak{W}_0 \rightarrow \mathbb{Z}/2\mathbb{Z}.$$

Composing it with the map $q_V : \mathfrak{W}_V \rightarrow \mathfrak{W}_0$, we get the dimension homomorphism

$$\dim : \mathfrak{W}_V \rightarrow \mathbb{Z}/2\mathbb{Z}.$$

Case 1: $U(V)$ is an orthogonal group.

Lemma 3.6. *If $U(V)$ is an orthogonal group, then there is an exact sequence*

$$1 \rightarrow \{1, \text{sgn}\} \rightarrow \mathfrak{W}_V = \text{Hom}(O(V), \mathbb{C}^\times) \xrightarrow{\nu_V} \widehat{K} = \widehat{F^\times/(F^\times)^2} \rightarrow 1.$$

For every $\chi \in \text{Hom}(O(V), \mathbb{C}^\times)$, $\nu_V(\chi)$ equals the descending of $\chi|_{SO(V)}$ through the spinor norm map $SO(V) \rightarrow F^\times/(F^\times)^2$.

Proof. For every non-isotropic vector $v \in V$, denote by

$$s_v : V \rightarrow V, \quad u \mapsto u - \frac{2\langle u, v \rangle_V}{\langle v, v \rangle} u$$

the reflection along v . Recall that there is a unique homomorphism

$$(33) \quad \delta_V : O(V) \rightarrow F^\times/(F^\times)^2$$

which maps every reflection s_v to the class of $\langle v, v \rangle_V$. The restriction of δ_V to $SO(V)$ is called the spinor norm map. As V is not anisotropic, the homomorphism

$$\text{sgn} \times \delta_V : O(V) \rightarrow \{\pm 1\} \times (F^\times/(F^\times)^2)$$

is surjective and its kernel is the commutator subgroup of $O(V)$. Therefore its pullback yields a group isomorphism

$$\text{Hom}(O(V), \mathbb{C}^\times) \cong \{\pm 1\} \times \widehat{F^\times/(F^\times)^2}.$$

The lemma now follows by noting that the diagram

$$\begin{array}{ccc} U(V)_{\text{split}} & \xrightarrow{\text{inclusion}} & SO(V) \\ \downarrow \mu_V & & \downarrow \delta_V \\ F^\times/(F^\times)^2 & \xlongequal{\quad} & F^\times/(F^\times)^2 \end{array}$$

commutes. □

Case 2: $U(V)$ is a symplectic group and $F \not\cong \mathbb{C}$.

Recall the discriminant map (for orthogonal spaces)

$$\begin{aligned} \text{disc} : \mathfrak{W}_0 &\rightarrow F^\times/(F^\times)^2, \\ (W, \langle, \rangle_W) &\mapsto (-1)^{\frac{m(m-1)}{2}} \det[\langle e_i, e_j \rangle_W]_{1 \leq i, j \leq m}, \end{aligned}$$

where e_1, e_2, \dots, e_m is a basis of W . Denote by H the division quaternion algebra over F , equipped with the orthogonal form

$$(x, y) \mapsto \text{the reduced trace of } x\bar{y},$$

where “ $-$ ” stands for the main involution of H . By abuse of notation, we still use H to denote its Witt class. Then as an element of the Witt group \mathfrak{W}_0 , H has trivial discriminant, has order 2 if F is non-archimedean, and has infinite order if $F = \mathbb{R}$.

Lemma 3.7. *If $U(V)$ is a symplectic group and $F \not\cong \mathbb{C}$, then we have an exact sequence*

$$1 \rightarrow \langle H \rangle \rightarrow \mathfrak{W}_V = \mathfrak{W}_0 \xrightarrow{\nu_V} \widehat{\widehat{K}} \rightarrow 1.$$

For every $\sigma \in \mathfrak{W}_0$, if $\dim \sigma$ is even, then

$$\nu_V(\sigma)(a, t) = (\text{disc}(\sigma), a)_F, \quad (a, t) \in \widetilde{K} = F^\times / (F^\times)^2 \times \{\pm 1\};$$

if $\dim \sigma$ is odd, then

$$\nu_V(\sigma) = \gamma_{\text{disc}(\sigma)\psi}.$$

Proof. In view of the equality (29), this is an easy consequence of Lemma 3.4. \square

Case 3: $U(V)$ is a unitary group.

In this case, \mathfrak{W}_0 is isomorphic to \mathbb{Z} if $F = \mathbb{R}$, and has order 4 if F is non-archimedean. Define a homomorphism

$$\widehat{K} = \text{Hom}(E^\times / \{xx' \mid x \in E^\times\}, \mathbb{C}^\times) \xrightarrow{\text{restriction}} \text{Hom}(F^\times / \{xx' \mid x \in E^\times\}, \mathbb{C}^\times) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}.$$

Lemma 3.8. *If $U(V)$ is a unitary group, then the following diagram of abelian groups is Cartesian:*

$$\begin{array}{ccc} \mathfrak{W}_V & \xrightarrow{\nu_V} & \widehat{K} \\ \downarrow q_V & & \downarrow \text{restriction} \\ \mathfrak{W}_0 & \xrightarrow{\dim} & \mathbb{Z}/2\mathbb{Z}. \end{array}$$

Consequently we have an exact sequence

$$1 \rightarrow \mathfrak{W}_0^{\text{even}} \rightarrow \mathfrak{W}_V \xrightarrow{\nu_V} \widehat{K} \rightarrow 1,$$

where $\mathfrak{W}_0^{\text{even}}$ is the kernel of the homomorphism $\dim : \mathfrak{W}_0 \rightarrow \mathbb{Z}/2\mathbb{Z}$.

Proof. It follows from the discussion of [HKS, Section 1] that the image of

$$(34) \quad \nu_V \times q_V : \mathfrak{W}_V \rightarrow \widehat{K} \times \mathfrak{W}_0$$

is contained in the fibre product $\widehat{K} \times_{\mathbb{Z}/2\mathbb{Z}} \mathfrak{W}_0$.

Put

$$U(E) := \{x \in E^\times \mid xx' = 1\}.$$

Then pullback through the determinant homomorphism $\det : U(V) \rightarrow U(E)$ induces an identification

$$\Omega_V(0) = \text{Hom}(U(V), \mathbb{C}^\times) = \text{Hom}(U(E), \mathbb{C}^\times).$$

Hilbert's Theorem 90 says that the map

$$E^\times/F^\times \rightarrow U(E), \quad x \mapsto \frac{x}{x^t}$$

is a topological isomorphism. Therefore we have a further identification

$$\Omega_V(0) = \widehat{U(E)} = \widehat{E^\times/F^\times}.$$

Note that E^\times/F^\times is a quotient of K , and $\widehat{E^\times/F^\times}$ is a subgroup of \widehat{K} of index two. It is easy to check that the restriction of ν_V to $\Omega_V(0)$ equals the inclusion map $\widehat{E^\times/F^\times} \hookrightarrow \widehat{K}$. This implies that (34) is injective and the image of (34) contains $\widehat{E^\times/F^\times}$.

Consider the exact sequence

$$1 \rightarrow \widehat{E^\times/F^\times} \rightarrow \widehat{K} \times_{\mathbb{Z}/2\mathbb{Z}} \mathfrak{W}_0 \xrightarrow{p_0} \mathfrak{W}_0 \rightarrow 1,$$

where p_0 is the projection map. Note that the restriction of p_0 to the image of (34) is surjective. Therefore the image of (34) equals $\widehat{K} \times_{\mathbb{Z}/2\mathbb{Z}} \mathfrak{W}_0$ since it already contains $\widehat{E^\times/F^\times}$. This finishes the proof. \square

Case 4: $U(V)$ is a non-archimedean quaternionic symplectic group.

Recall the discriminant map (for quaternionic skew-Hermitian spaces)

$$\begin{aligned} \text{disc} : \mathfrak{W}_0 &\rightarrow F^\times/(F^\times)^2, \\ (W, \langle \cdot, \cdot \rangle_W) &\mapsto (-1)^m \det[\langle e_i, e_j \rangle_W]_{1 \leq i, j \leq m}, \end{aligned}$$

where e_1, e_2, \dots, e_m is a basis of W .

Lemma 3.9. *If $U(V)$ is a non-archimedean quaternionic symplectic group, then we have an exact sequence*

$$1 \rightarrow \{1, D_-^3\} \rightarrow \mathfrak{W}_V = \mathfrak{W}_0 \xrightarrow{\nu_V} \widehat{K} = F^\times/(F^\times)^2 \rightarrow 1,$$

where D_-^3 is the element of \mathfrak{W}_0 represented by the 3-dimensional anisotropic skew-Hermitian left D -vector space. For every $\sigma \in \mathfrak{W}_0$, we have

$$\nu_V(\sigma) = (\text{disc}(\sigma), \cdot)_F.$$

Proof. See [Ya, Section 6]. \square

Case 5: $U(V)$ is a non-archimedean quaternionic orthogonal group.

In this case, there is a unique homomorphism $U(V) \rightarrow F^\times/(F^\times)^2$ which extends the map $\mu_V : U(V)_{\text{split}} \rightarrow K = F^\times/(F^\times)^2$ (see [Wa, Section 2]). We still denote it

by μ_V . It also follows from [Wa, Section 2] that every character of $U(V)$ uniquely descends to a character of $F^\times/(F^\times)^2$. Therefore the homomorphism

$$(\nu_V)|_{\Omega_V(0)} : \Omega_V(0) \rightarrow \widehat{F^\times/(F^\times)^2},$$

which equals the descending of characters through $\mu_V : U(V) \rightarrow F^\times/(F^\times)^2$, is an isomorphism.

Now the following lemma is clear.

Lemma 3.10. *If $U(V)$ is a non-archimedean quaternionic orthogonal group, then*

$$\nu_V \times \dim : \mathfrak{W}_V \xrightarrow{\sim} \widehat{F^\times/(F^\times)^2} \times \mathbb{Z}/2\mathbb{Z}.$$

is a group isomorphism.

Case 6: $U(V)$ is a real quaternionic orthogonal group.

In this case both $\Omega_V(0)$ and K are trivial, and we have

Lemma 3.11. *If $U(V)$ is a real quaternionic orthogonal group, then $\mathfrak{W}_V = \mathfrak{W}_0 \cong \mathbb{Z}$ and $K = \{1\}$.*

Case 7: $U(V)$ is a complex symplectic group or a real quaternionic symplectic group.

In this case both $\Omega_V(0)$ and K are also trivial, and we have

Lemma 3.12. *If $U(V)$ is a complex symplectic group or a real quaternionic symplectic group, then $\mathfrak{W}_V = \mathfrak{W}_0 \cong \mathbb{Z}/2\mathbb{Z}$ and $K = \{1\}$.*

To summarize, we have the following

Proposition 3.13. *The homomorphism $\nu_V : \mathfrak{W}_V \rightarrow \widehat{K'}$ is surjective, and its kernel is*

$$\text{Ker}(\nu_V) \cong \begin{cases} \mathbb{Z}, & \text{if } U(V) \text{ is a real symplectic group, a real unitary group,} \\ & \text{or a real quaternionic orthogonal group;} \\ \mathbb{Z}/2\mathbb{Z}, & \text{otherwise.} \end{cases}$$

3.4. The Kudla kernel. Now assume that V is arbitrary. Let V' be a nonzero split ϵ -Hermitian right D -vector space so that V is isometrically isomorphic to a subspace of V' . For every $\kappa \in \widehat{K'}$, put

$$(35) \quad \mathfrak{W}_{V,\kappa} := r_V^{V'}(\nu_{V'}^{-1}(\kappa)).$$

By Lemma 3.5, we know that $\mathfrak{W}_{V,\kappa}$ is independent of V' . The Kudla kernel in \mathfrak{W}_V is then

$$(36) \quad \mathcal{K}_V := \mathfrak{W}_{V,\kappa_0},$$

where κ_0 is the trivial character of K' .

Lemma 3.14. *The restriction map $\mathbf{r}_V^{V'} : \mathcal{K}_{V'} \rightarrow \mathcal{K}_V$ is an isomorphism unless V is the zero orthogonal space. Consequently, we have*

$$(37) \quad \mathcal{K}_V \cong \begin{cases} 0, & \text{U}(V) \text{ is the trivial orthogonal group;} \\ \mathbb{Z}, & \text{U}(V) \text{ is a real symplectic group, a real unitary group,} \\ & \text{or a real quaternionic orthogonal group;} \\ \mathbb{Z}/2\mathbb{Z}, & \text{otherwise.} \end{cases}$$

Proof. The first part of the lemma is equivalent to the following: if $\mathrm{U}(V)$ is not the trivial orthogonal group, then every character on $\mathrm{U}(V')$ is trivial if it has trivial Kudla character and trivial restriction to $\mathrm{U}(V)$. This follows by the argument of Section 3.3. The second part follows from the first one and Proposition 3.13. \square

4. DEGENERATE PRINCIPAL SERIES AND THE DOUBLING METHOD

Fix an element $\kappa \in \widehat{K}'$ throughout this section. Denote by $\Omega_{V,\kappa}$ the inverse image of $\mathfrak{W}_{V,\kappa}$ under the quotient map $\Omega_V \rightarrow \mathfrak{W}_V$.

4.1. Two results on degenerate principal series representations. Assume that V is split of rank $r > 0$, with a Lagrangian subspace X . By abuse of notation, still write $\kappa := \kappa \circ \mu'_V \in \mathrm{Hom}(\mathrm{U}'(V)_{\mathrm{split}}, \mathbb{C}^\times)$. For $s \in \mathbb{C}$, define the following normalized degenerate principal series representation of $\mathrm{U}'(V)$:

$$(38) \quad \mathrm{I}_\kappa(s) := \{f \in C^\infty(\mathrm{U}'(V)) \mid f(px) = \kappa(p)|p|_X^{s+\rho_r} f(x), p \in \mathrm{P}'(X), x \in \mathrm{U}'(V)\},$$

where ρ_r is as in (3), and $|\cdot|_X$ is as in (20). Under right translations, this is a smooth representation of $\mathrm{U}'(V)$.

For every $\sigma = (W, \omega) \in \Omega_{V,\kappa}$, the functional $\lambda_{X \otimes_{\mathbb{D}} W}$ induces a $\mathrm{U}'(V)$ -intertwining map

$$(39) \quad \begin{aligned} \Psi : \omega &\rightarrow \mathrm{I}_\kappa\left(\frac{\dim \sigma}{2} - \rho_r\right), \\ v &\mapsto (g \mapsto \lambda_{X \otimes_{\mathbb{D}} W}(g \cdot v)). \end{aligned}$$

See equation (21). Denote by R_σ the image of Ψ (equipped with the quotient topology in the archimedean case):

$$(40) \quad \mathrm{R}_\sigma := \Psi(\omega) \subseteq \mathrm{I}_\kappa\left(\frac{\dim \sigma}{2} - \rho_r\right).$$

If we extend ω to a representation of $(\mathrm{U}'(V) \times \mathrm{U}(W)) \ltimes \mathrm{H}_V(W)$ so that $\lambda_{X \otimes_{\mathbb{D}} W}$ is $\mathrm{U}(W)$ -invariant, then R_σ is the maximal (Hausdorff in the archimedean case) quotient of ω on which $\mathrm{U}(W)$ acts trivially. See [Ra1], [MVW] (non-archimedean), and [KR1], [Zh] (archimedean).

Lemma 4.1. *For every $\sigma \in \Omega_{V,\kappa}$ with the split rank $\mathrm{rank} \sigma \geq r$, we have that*

$$\mathrm{R}_\sigma = \mathrm{I}_\kappa\left(\frac{\dim \sigma}{2} - \rho_r\right).$$

Proof. The assertion follows from the work of Kudla-Rallis [KR2], Kudla-Sweet [KS], and Yamana [Ya] (non-archimedean), and Lee-Zhu [LZ1, LZ2, LZ3, LZ4], Lee [LL], and Yamana [Ya] (archimedean). \square

The first key point of this article is the following proposition, which is responsible for the upper bound in conservation relations.

Proposition 4.2. *Let $m \geq 2\rho_r$ be an integer.*

- (i) *If $U(V)$ is neither a complex symplectic group nor a quaternionic group, then for every $\mathbf{t} \in \mathfrak{W}_{V,\kappa}$ such that $\dim \mathbf{t} \equiv m \pmod{2}$, we have that*

$$\frac{I_\kappa\left(\frac{m}{2} - \rho_r\right)}{\sum_{\mathbf{t}' \in \mathfrak{W}_{V,\kappa}, \mathbf{t}' \neq \mathbf{t}; \sigma' \in \mathbf{t}', \dim \sigma' = m} R_{\sigma'}} \cong \begin{cases} R_\sigma, & \text{if there exists an element } \sigma \in \mathbf{t} \text{ of dimension } 4\rho_r - m; \\ 0, & \text{otherwise} \end{cases}$$

as $U'(V)$ -representations.

- (ii) *If $U(V)$ is a non-archimedean quaternionic group or a real quaternionic orthogonal group, then as $U(V)$ -representations,*

$$\frac{I_\kappa\left(\frac{m}{2} - \rho_r\right)}{\sum_{\mathbf{t}' \in \mathfrak{W}_{V,\kappa}; \sigma' \in \mathbf{t}', \dim \sigma' = m} R_{\sigma'}} \cong \bigoplus_{\mathbf{t} \in \mathfrak{W}_{V,\kappa}; \sigma \in \mathbf{t}, \dim \sigma = 4\rho_r - m} R_\sigma.$$

Proof. The assertion can be read off from a number of results in the literature. For non-archimedean cases, the relevant results are in Kudla-Rallis [KR2, Introduction], Kudla-Sweet [KS, Theorem 1.2], and Yamana [Ya, Introduction]. For archimedean cases, the relevant results are in Lee-Zhu [LZ1, Introduction], [LZ2, Section 4], [LZ3, Theorem 1], [LZ4, Section 6], Lee [LL, Appendix], and Yamana [Ya, Section 9]. \square

4.2. The doubling method. Now we allow V to be non-split. Put $\mathbb{V} := V \oplus V^-$ throughout this article, and note that $\Delta := \{(v, v) \mid v \in V\}$ is a Lagrangian subspace of \mathbb{V} . Still assume that V is nonzero. For every $s \in \mathbb{C}$, we form the (normalized) degenerate principal series representation $I_\kappa(s)$ of $U'(\mathbb{V})$ as in the last subsection. Note that there is a natural homomorphism $U'(V) \times U'(V^-) \rightarrow U'(\mathbb{V})$.

Let $\pi \in \text{Irr}(U'(V))$ be an irreducible representation which is genuine with respect to some (and hence all) elements of $\mathfrak{W}_{V,\kappa}$. The theory of local Zeta integrals [PSR, LR] implies that

Lemma 4.3. *For every $s \in \mathbb{C}$,*

$$\text{Hom}_{U'(V)}(I_\kappa(s), \pi) \neq 0.$$

The following is a trivial criterion for non-vanishing of theta liftings:

Lemma 4.4. *Let $\sigma_\mathbb{V} \in \Omega_{\mathbb{V},\kappa}$ and denote $\sigma_V := r_V^\mathbb{V}(\sigma_\mathbb{V})$. Then*

$$\text{Hom}_{U'(V)}(R_{\sigma_V}, \pi) \neq 0 \quad \text{implies} \quad \Theta_{\sigma_V}(\pi) \neq 0.$$

On the other hand, we have

Lemma 4.5. *Let $\sigma_{\mathbb{V}} = (W, \omega) \in \Omega_{\mathbb{V}, \kappa}$ and denote $\sigma_V := r_V^{\mathbb{V}}(\sigma_{\mathbb{V}})$. If κ is trivial and W is anisotropic, then*

$$\Theta_{\sigma_V}(\pi) \neq 0 \quad \text{implies} \quad \text{Hom}_{U'(V) \times U'(V^-)}(R_{\sigma_V}, \pi \widehat{\otimes} \pi^{\vee}) \neq 0.$$

Proof. As in (15), write

$$\omega|_{J'_V(W) \times J'_{V^-}(W)} = \omega|_V \widehat{\otimes} \omega|_{V^-}, \quad \omega|_V \in \Omega_V(W), \quad \omega|_{V^-} \in \Omega_{V^-}(W).$$

The triviality of κ implies that $\omega|_V$ and $\omega|_{V^-}$ are contragredient to each other with respect to the isomorphism

$$\begin{aligned} U'(V) \ltimes ((V \otimes_{\mathbb{D}} W) \times F) &\rightarrow U'(V^-) \ltimes ((V^- \otimes_{\mathbb{D}} W) \times F), \\ (g; u, t) &\mapsto (g; u, -t). \end{aligned}$$

Extends $\omega|_V$ and $\omega|_{V^-}$ to representations of $J'_{V,W}$ and $J'_{V^-,W}$, respectively, so that they are contragredient to each other with respect to the isomorphism

$$\begin{aligned} (U'(V) \times U'(W)) \ltimes ((V \otimes_{\mathbb{D}} W) \times F) &\rightarrow (U'(V^-) \times U'(W)) \ltimes ((V^- \otimes_{\mathbb{D}} W) \times F), \\ (g, h; u, t) &\mapsto (g, h; u, -t). \end{aligned}$$

Assume that $\Theta_{\sigma_V}(\pi) \neq 0$. Then there is an irreducible representation $\tau \in \text{Irr}(U'(W))$ such that

$$(41) \quad \text{Hom}_{U'(V) \times U'(W)}(\omega|_V, \pi \widehat{\otimes} \tau) \neq 0.$$

Since W is anisotropic, both π and τ are unitarizable. By taking complex conjugations on the representations in (41), we have that

$$(42) \quad \text{Hom}_{U'(V^-) \times U'(W)}(\omega|_{V^-}, \pi^{\vee} \widehat{\otimes} \tau^{\vee}) \neq 0.$$

Combining (41) and (42), we have that

$$\text{Hom}_{U'(V) \times U'(V^-)}((\omega|_V \widehat{\otimes} \omega|_{V^-})_{U'(W)}, \pi \widehat{\otimes} \pi^{\vee}) \neq 0,$$

where a subscript “ $U'(W)$ ” indicates the maximal (Hausdorff in the archimedean case) quotient on which $U'(W)$ acts trivially. The lemma then follows by noting that

$$(\omega|_V \widehat{\otimes} \omega|_{V^-})_{U'(W)} \cong R_{\sigma_V},$$

as representations of $U'(V) \times U'(V^-)$. □

Remark: The lemma above is a variant of a more well-known result in the literature on local theta correspondence ([HKS, Proposition 3.1] and [Ku2, Proposition 1.5]). Note that we include the non-archimedean quaternionic case (for which MVW-involutions do not exist). To compensate this, the space W is assumed to be anisotropic, which is what we need (for Lemma 5.3).

5. NON-OCCURRENCE OF THE TRIVIAL REPRESENTATION BEFORE STABLE RANGE

Write $d := \dim_{\mathbb{D}} V \geq 0$ for simplicity. The purpose of this section is to show the following proposition, which is the second key point of this article and which is responsible for the lower bound in conservation relations.

Proposition 5.1. *If F is non-archimedean or $U(V)$ is an orthogonal group, then*

$$n_{\mathbf{t}_V}(1_V) = 4\rho_d + 2.$$

If $K_V \cong \mathbb{Z}$ (that is, $U(V)$ is a real symplectic group, a real unitary group, or a real quaternionic orthogonal group), then for every nonzero element $\mathbf{t} \in K_V$,

$$n_{\mathbf{t}}(1_V) = 4\rho_d + 2 + 4\rho_1(|\mathbf{t}| - 1).$$

Recall that when F is non-archimedean or $U(V)$ is an orthogonal group, \mathbf{t}_V denotes the anti-split Witt tower, which is the unique generator of K_V ; and if $\mathbf{t} \in K_V \cong \mathbb{Z}$, $|\mathbf{t}|$ denotes the nonnegative integer so that \mathbf{t} is $|\mathbf{t}|$ -multiple of a generator of K_V . As noted in the Introduction, we have $\deg \mathbf{t}_V = 4\rho_1$, and for the latter case $\deg \mathbf{t} = 4\rho_1|\mathbf{t}|$. See the explicit description of K_V in Sections 3.3 and 3.4.

Proposition 5.1 is trivial when $V = 0$. We assume that V is nonzero in the rest of this section. In view of Kudla's persistence principle, Proposition 5.1 is clearly equivalent to the following

Proposition 5.2. *Assume that F is non-archimedean, or $U(V)$ is a real or complex orthogonal group, a real symplectic group, a real unitary group, or a real quaternionic orthogonal group. Let \mathbf{t} be a nonzero element of K_V , and denote by σ_{d-1} the element of \mathbf{t} of split rank $d - 1$. Then*

$$\Theta_{\sigma_{d-1}}(1_V) = 0.$$

All archimedean cases of Proposition 5.2 are proved in [Pr2, Appendix C], [Pa2, Lemma 3.1], [LPTZ, Proposition 3.38], and [LL, Theorem 1.2.1], and they follow easily from the correspondence of K -types in the space of joint harmonics [Ho3], where K stands for a maximal compact subgroup.

For non-archimedean cases, parts of results are in [Ra1, Appendix] (for orthogonal groups), [KR3, Lemma 4.2] (for symplectic groups), and [GG, Theorem 2.9] (for unitary groups). Only the non-archimedean quaternionic case is new. Because of the lack of MVW-involutions, the approach of [KR3] and [GG], which uses the doubling method, does not work for this case. In the remainder of this section, we follow the idea of Rallis ([Ra1, Ra2], which treat the case of orthogonal groups) to provide a uniform proof of Proposition 5.2 in all non-archimedean cases.

5.1. The result for the non-archimedean case. Now F is non-archimedean and $d > 0$. We introduce the following notation concerning function spaces. For every totally disconnected locally compact Hausdorff topological space M , recall

that $C^\infty(M)$ denotes the space of (complex valued) locally constant functions on M . Denote by $C^\varsigma(M) \subset C^\infty(M)$ the subspace compactly support functions. If M is a locally analytic manifold over F , denote by $D^\varsigma(M)$ and $D_{1/2}^\varsigma(M)$ the space of Schwartz densities and Schwartz half densities on M , respectively. If μ is an everywhere positive smooth measure on M , then

$$D^\varsigma(M) = C^\varsigma(M)\mu \quad \text{and} \quad D_{1/2}^\varsigma(M) = C^\varsigma(M)\mu^{\frac{1}{2}}.$$

A linear functional on $D^\varsigma(M)$ is called a generalized function on M . Denote by $C^{-\infty}(M)$ the space of generalized functions on M . It contains $C^\infty(M)$ as a subspace. If M carries a locally analytic action of a group G , then the spaces $D^\varsigma(M)$, $D_{1/2}^\varsigma(M)$ and $C^{-\infty}(M)$ all carry natural G -actions.

Write $\sigma_V = (W^\circ, \omega_V)$ for the element of \mathfrak{t}_V of minimal dimension (W° is the anisotropic space of dimension $4\rho_1$). We view ω_V as a representation of $U(V)$ (when $U'(V)$ is a metaplectic group, the restriction of ω_V to $U'(V)$ descends to a representation of $U(V)$). Likewise, view 1_V as the trivial representation of $U(V)$.

Then Proposition 5.2 amounts to saying that

$$(43) \quad \text{Hom}_{U(V)}(D_{1/2}^\varsigma(V^{d-1}) \otimes \omega_V, 1_V) = 0,$$

where V^{d-1} carries the diagonal action of $U(V)$. Since the Haar measure on V^{d-1} is $U(V)$ -invariant, the validity of (43) does not change if we replace $D_{1/2}^\varsigma(V^{d-1})$ by $D^\varsigma(V^{d-1})$ or $C^\varsigma(V^{d-1})$.

We argue by induction on d , and assume that (43) holds when d is smaller.

5.2. Non-occurrence of 1_V in ω_V . We start with the following observation:

Lemma 5.1. *Let ω be a smooth oscillator representation of $J_V(W)$. If V is split with a Lagrangian subspace X , and W is anisotropic. Then every linear functional on ω is $N(X)$ -invariant if and only if it is invariant under $X \otimes_{\mathbb{D}} W \subset H_V(W)$, where $N(X)$ denote the unipotent radical of $P(X)$.*

Proof. This follows by using the Schrodinger model of an oscillator representation ([Ho2, Part II, Section 3]). \square

Lemma 5.2. *Let ω be a smooth oscillator representation of $J_V(W)$. If V is split and nonzero, and W is anisotropic and nonzero. Then*

$$\text{Hom}_{U(V)}(\omega, 1_V) = 0.$$

Proof. Let X be a Lagrangian subspace of V . Assume that there is a nonzero element $\lambda \in \text{Hom}_{U(V)}(\omega, 1_V)$. Then Lemma 5.1 implies that λ is a scalar multiple of $\lambda_{X \otimes_{\mathbb{D}} W}$. This contradicts the equality (21), as all Kudla characters are unitary. \square

Lemma 5.3. *If $d = 1$, then*

$$(44) \quad \text{Hom}_{U(V)}(\omega_V, 1_V) = 0.$$

Proof. Introduce the space $\mathbb{V} := V \oplus V^-$ as in Section 4.2, which is split of rank 1. Write $\sigma_{\mathbb{V}}$ for the element of $\mathfrak{t}_{\mathbb{V}}$ of minimal dimension ($= 4\rho_1$). Then by Lemma 3.14, $r_V^{\mathbb{V}}(\sigma_{\mathbb{V}}) = \sigma_V$.

As an simple instance of Proposition 4.2, we have an exact sequence of representations of $U(\mathbb{V})$:

$$(45) \quad 0 \rightarrow R_{\sigma_{\mathbb{V}}} \rightarrow I_{\kappa_0}(\rho_1) \rightarrow 1_{\mathbb{V}} \rightarrow 0,$$

where κ_0 denotes the trivial Kudla character. Note that

$$I_{\kappa_0}(\rho_1) \cong D^s(P(\Delta) \backslash U(\mathbb{V}))$$

as representations of $U(\mathbb{V})$. When restricted to $U(V) \times U(V^-)$,

$$D^s(P(\Delta) \backslash U(\mathbb{V})) = D^s(U(V)^{\Delta} \backslash (U(V) \times U(V^-))),$$

where

$$U(V)^{\Delta} := (U(V) \times U(V^-)) \cap P(\Delta)$$

is the group $U(V)$ diagonally embedded in $U(V) \times U(V^-)$. We have a $U(V) \times U(V^-)$ -stable decomposition

$$D^s(U(V)^{\Delta} \backslash (U(V) \times U(V^-))) = D_0^s(U(V)^{\Delta} \backslash (U(V) \times U(V^-))) \oplus \mathbb{C}\mu_0,$$

where D_0^s stands for the space of Schwartz densities of total volume 0, and μ_0 is a right invariant positive measure on $U(V)^{\Delta} \backslash U(V) \times U(V^-)$. Then uniqueness of invariant generalized functions together with the exact sequence (45) implies that

$$R_{\sigma_{\mathbb{V}}} \cong D_0^s(U(V)^{\Delta} \backslash (U(V) \times U(V^-))),$$

and furthermore,

$$\text{Hom}_{U(V) \times U(V^-)}(R_{\sigma_{\mathbb{V}}}, 1_V \otimes 1_{V^-}) = 0.$$

This proves the result by Lemma 4.5. □

Lemma 5.4. *One has that*

$$(46) \quad \text{Hom}_{U(V)}(\omega_V, 1_V) = 0.$$

Proof. When V is a symplectic space, this is a special case of Lemma 5.2. Now assume that V is not a symplectic space. Then there is an orthogonal decomposition of ϵ -Hermitian spaces $V = V_1 \oplus V_2$ such that $\dim_{\mathbb{D}} V_1 = 1$. Then

$$\text{Hom}_{U(V)}(\omega_V, 1_V) \subset \text{Hom}_{U(V_1)}(\omega_V, 1_{V_1}) = \text{Hom}_{U(V_1)}(\omega_{V_1} \otimes \omega_{V_2}, 1_{V_1}) = 0,$$

by Lemma 5.3. □

5.3. Reduction to the null cone. Put

$$\Gamma := \{(v_1, v_2, \dots, v_{d-1}) \in V^{d-1} \mid \langle v_i, v_j \rangle_V = 0, \forall i, j = 1, 2, \dots, d-1\},$$

the null cone in V^{d-1} . Pushing forward of Schwartz densities yields an inclusion

$$D^\varsigma(V^{d-1} \setminus \Gamma) \subset D^\varsigma(V^{d-1}).$$

Recall that (43) is assumed to hold when $d = \dim_{\mathbb{D}} V$ is smaller. The aim of this subsection is to prove

Lemma 5.5. *Every*

$$\lambda \in \text{Hom}_{\text{U}(V)}(D^\varsigma(V^{d-1}) \otimes \omega_V, 1_V)$$

vanishes on $D^\varsigma(V^{d-1} \setminus \Gamma) \otimes \omega_V$.

If $d = 1$ or if V is a symplectic space of dimension 2, then $\Gamma = V^{d-1}$ and Lemma 5.5 is trivial. So assume that either V is a symplectic space of dimension ≥ 4 , or V is not a symplectic space and $d \geq 2$.

Let V_0 be a nonzero non-degenerate subspace of V of minimal dimension d_0 . Thus $d_0 = 2$ if V is a symplectic space, and $d_0 = 1$ otherwise. Denote by V_0^\perp the orthogonal complement of V_0 in V . Put

$$B_0 := \begin{cases} \{(v_1, v_2) \in (V_0)^2 \mid v_1, v_2 \text{ is a basis of } V_0\}, & \text{if } V \text{ is a symplectic space;} \\ V_0 \setminus \{0\}, & \text{otherwise.} \end{cases}$$

Then

$$S_0 := B_0 \times V^{d-d_0-1} \subset V^{d-1}$$

is stable under $\text{U}(V_0^\perp) \subset \text{U}(V)$, and the map

$$(47) \quad \Phi : \text{U}(V) \times S_0 \rightarrow V^{d-1}, \quad (g, \mathbf{v}) \mapsto g \cdot \mathbf{v}$$

is a $\text{U}(V) \times \text{U}(V_0^\perp)$ -equivariant submersion, where $\text{U}(V) \times \text{U}(V_0^\perp)$ acts on $\text{U}(V) \times S_0$ by

$$(g, h) \cdot (x, \mathbf{v}) := (gxh^{-1}, h \cdot \mathbf{v}),$$

and acts on V^{d-1} by

$$(g, h) \cdot \mathbf{v} := g \cdot \mathbf{v}.$$

Lemma 5.6. *One has that*

$$(48) \quad \text{Hom}_{\text{U}(V) \times \text{U}(V_0^\perp)}(D^\varsigma(\text{U}(V) \times S_0) \otimes \omega_V, 1_V) = 0,$$

where we extend the representations ω_V and 1_V of $\text{U}(V)$ to $\text{U}(V) \times \text{U}(V_0^\perp)$ by the trivial action of $\text{U}(V_0^\perp)$.

Proof. Frobenius reciprocity [Be, Section 1.5] implies that the left hand side of (48) equals

$$(49) \quad \text{Hom}_{\text{U}(V_0^\perp)}(D^\varsigma(S_0) \otimes (\omega_V)|_{\text{U}(V_0^\perp)}, 1_{V_0^\perp}).$$

Note that

$$D^\varsigma(S_0) = D^\varsigma(B_0 \times V_0^{d-d_0-1}) \otimes D^\varsigma((V_0^\perp)^{d-d_0-1}),$$

and

$$\omega_V = \omega_{V_0} \otimes \omega_{V_0^\perp}.$$

By the induction assumption, we have

$$\mathrm{Hom}_{\mathrm{U}(V_0^\perp)}(D^\varsigma((V_0^\perp)^{d-d_0-1}) \otimes \omega_{V_0^\perp}, 1_{V_0^\perp}) = 0,$$

and therefore the space (49) vanishes as well. \square

Since the map Φ is a submersion, its image $\mathrm{U}(V) \cdot S_0$ is open in V^{d-1} . We have the following

Lemma 5.7. *Every element $\lambda \in \mathrm{Hom}_{\mathrm{U}(V)}(D^\varsigma(V^{d-1}) \otimes \omega_V, 1_V)$ vanishes on*

$$D^\varsigma(\mathrm{U}(V) \cdot S_0) \otimes \omega_V.$$

Proof. Pushing forward through Φ yields a $\mathrm{U}(V) \times \mathrm{U}(V_0^\perp)$ -intertwining linear map

$$\Phi_* : D^\varsigma(\mathrm{U}(V) \times S_0) \rightarrow D^\varsigma(V^{d-1}).$$

Tensoring with ω_V , we get a $\mathrm{U}(V) \times \mathrm{U}(V_0^\perp)$ -intertwining linear map

$$\Phi_* \otimes 1_{\omega_V} : D^\varsigma(\mathrm{U}(V) \times S_0) \otimes \omega_V \rightarrow D^\varsigma(V^{d-1}) \otimes \omega_V.$$

Since both $D^\varsigma(V^{d-1}) \otimes \omega_V$ and 1_V carries the trivial $\mathrm{U}(V_0^\perp)$ -action, we have

$$\lambda \in \mathrm{Hom}_{\mathrm{U}(V) \times \mathrm{U}(V_0^\perp)}(D^\varsigma(V^{d-1}) \otimes \omega_V, 1_V).$$

Now Lemma 5.6 implies that $\lambda \circ (\Phi_* \otimes 1_{\omega_V}) = 0$, namely, λ vanishes on the image of $\Phi_* \otimes 1_{\omega_V}$, which is $D^\varsigma(\mathrm{U}(V) \cdot S_0) \otimes \omega_V$. \square

We are now ready to prove Lemma 5.5. Let the group $\mathrm{GL}_{d-1}(\mathbb{D})$ act on V^{d-1} by

$$g \cdot (v_1, v_2, \dots, v_{d-1}) := (v_1, v_2, \dots, v_{d-1})g^{-1}.$$

This commutes with the action of $\mathrm{U}(V)$. Lemma 5.7 implies that every $\lambda \in \mathrm{Hom}_{\mathrm{U}(V)}(D^\varsigma(V^{d-1}) \otimes \omega_V, 1_V)$ vanishes on

$$D^\varsigma((\mathrm{U}(V) \times \mathrm{GL}_{d-1}(\mathbb{D})) \cdot S_0) \otimes \omega_V.$$

Lemma 5.5 follows by noting that

$$\bigcup_{V_0} (\mathrm{U}(V) \times \mathrm{GL}_{d-1}(\mathbb{D})) \cdot S_0 = V^{d-1} \setminus \Gamma,$$

where V_0 runs through all non-degenerate subspaces of V of dimension d_0 .

5.4. **Vanishing on small orbits in the null cone.** Write

$$\Gamma := \bigsqcup_{i=0}^{\text{rank } V} \Gamma_i,$$

where

$\Gamma_i := \{(v_1, v_2, \dots, v_{d-1}) \in \Gamma \mid v_1, v_2, \dots, v_{d-1} \text{ span a } D\text{-subspace of } V \text{ of dimension } i\}.$

Then each Γ_i is a single $U(V) \times GL_{d-1}(D)$ -orbit.

Denote by \mathcal{O}_D the ring of integers in D :

$$\mathcal{O}_D := \{a \in D \mid |\det a|_E \leq 1\}.$$

Lemma 5.8. *For each $i = 0, 1, \dots, \text{rank } V$, Γ_i is a homogeneous space for the action of $U(V) \times GL_{d-1}(\mathcal{O}_D)$, and carries a $U(V) \times GL_{d-1}(\mathcal{O}_D)$ -invariant positive smooth measure μ_{Γ_i} on it.*

Proof. Fix a decomposition

$$V := (e_1 D \oplus e_2 D \oplus \dots \oplus e_i D) \oplus Y_i$$

such that $e_1 D \oplus e_2 D \oplus \dots \oplus e_i D$ is totally isotropic. Then the stabilizer of

$$\mathbf{v}_i = (e_1, e_2, \dots, e_i, 0, 0, \dots, 0) \in \Gamma_i$$

in $U(V) \times GL_{d-1}(D)$ is

$$S_i := \left\{ \left(\begin{bmatrix} a & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} a & 0 \\ * & * \end{bmatrix} \right) \in U(V) \times GL_{d-1}(D) \mid a \in GL_i(D) \right\}.$$

The first assertion of the lemma will follow from the equality

$$U(V) \times GL_{d-1}(D) = (U(V) \times GL_{d-1}(\mathcal{O}_D)) S_i.$$

This is an easy consequence of the Iwasawa decomposition:

$$GL_{d-1}(D) = GL_{d-1}(\mathcal{O}_D) \left\{ \begin{bmatrix} a & 0 \\ * & * \end{bmatrix} \in GL_{d-1}(D) \mid a \in GL_i(D) \right\}.$$

The second assertion of the lemma follows by noting that $(U(V) \times GL_{d-1}(\mathcal{O}_D)) \cap S_i$ is unimodular. \square

Lemma 5.9. *If $2i < d$, then*

$$\text{Hom}_{U(V)}(C^\infty(\Gamma_i) \otimes \omega_V, 1_V) = 0.$$

Proof. Fix an element

$$\mathbf{v} = (v_1, v_2, \dots, v_{d-1}) \in \Gamma_i.$$

Since $2i < d$, there is a nonzero non-degenerate subspace V_0 of V such that

$$\langle v, v_j \rangle_V = 0, \quad v \in V_0, \quad j = 1, 2, \dots, d-1.$$

Then \mathbf{v} is fixed by $U(V_0) \subset U(V)$, and the map

$$(50) \quad U(V) \times GL_{d-1}(V) \rightarrow \Gamma_i, \quad (g, h) \mapsto (g, h) \cdot \mathbf{v}$$

is a $U(V) \times U(V_0)$ -equivariant submersion, where $U(V) \times U(V_0)$ acts on $U(V) \times GL_{d-1}(V)$ by

$$(g, h) \cdot (x, y) := (gxh^{-1}, y),$$

and acts on Γ_i by

$$(g, h) \cdot x := g \cdot x.$$

As in the argument of last subsection, we have

$$\begin{aligned} & \text{Hom}_{U(V)}(C^\infty(\Gamma_i) \otimes \omega_V, 1_V) \\ \cong & \text{Hom}_{U(V)}(D^\infty(\Gamma_i) \otimes \omega_V, 1_V) \quad (\text{by Lemma 5.8}) \\ \hookrightarrow & \text{Hom}_{U(V) \times U(V_0)}(D^\infty(U(V) \times GL_{d-1}(V)) \otimes \omega_V, 1_V) \quad (\text{pushing forward of (50)}) \\ = & \text{Hom}_{U(V_0)}(D^\infty(GL_{d-1}(V)) \otimes (\omega_V)|_{U(V_0)}, 1_{V_0}) \quad (\text{by Frobenius reciprocity}) \\ = & \text{Hom}_{U(V_0)}(D^\infty(GL_{d-1}(V)) \otimes (\omega_{V_0^\perp}) \otimes \omega_{V_0}, 1_{V_0}) \\ = & 0 \quad (\text{by Lemma 5.4}). \end{aligned}$$

□

5.5. A homogeneity calculation for the main orbits in the null cone. In this subsection, assume that V splits. Write $d = 2r > 0$. View D^{d-1} as a right D -vector space of column vectors. For every subspace Z of D^{d-1} of dimension $r-1$, put

$$O_Z := \{\mathbf{v} \in \Gamma_r \mid \mathbf{v} \mathbf{a} = 0, \mathbf{a} \in Z\}.$$

By Witt's extension theorem, this is a single $U(V)$ -orbit, and we have

$$\Gamma_r = \bigsqcup_{Z \text{ is a subspace of } D^{d-1} \text{ of dimension } r-1} O_Z.$$

Let E^\times ($E = \text{center of } D$) act on V^{d-1} by

$$a \cdot x := xa^{-1}, \quad a \in E^\times, x \in V^{d-1}.$$

Then O_Z is E^\times -stable.

We will use the following convention throughout this article: if G is a group acting on two sets A and B , then for every $g \in G$ and every map $\varphi : A \rightarrow B$, $g \cdot \varphi : A \rightarrow B$ is the map defined by

$$(g \cdot \varphi)(a) := g \cdot (\varphi(g^{-1} \cdot a)), \quad a \in A.$$

If no action of G is specified on a set C , we consider C carries the trivial action of G .

Lemma 5.10. *The space $\text{Hom}_{U(V)}(\omega_V, C^\infty(O_Z))$ is one dimensional and every element λ of the space satisfies*

$$a \cdot \lambda = |a|_E^{-2\rho_1 d_B^2 r} \lambda, \quad a \in E^\times.$$

Proof. Fix an element $\mathbf{v}_Z = (v_1, v_2, \dots, v_{d-1}) \in O_Z$. Denote by X the Lagrangian subspace of V spanned by v_1, v_2, \dots, v_{d-1} . Fix a Lagrangian subspace Y of V which is complementary to X . For every $a \in E^\times$, denote by $m_a \in U(V)$ the element which stabilizes both X and Y , and acts on X through the scalar multiplication by a .

The stabilizer of \mathbf{v}_Z in $U(V)$ equals $N(X)$. Therefore

$$O_Z = U(V)/N(X).$$

The corresponding action of E^\times on $U(V)/N(X)$ is given by

$$a \cdot (gN(X)) = gm_a^{-1}N(X), \quad a \in E^\times.$$

By Frobenius reciprocity,

$$(51) \quad \text{Hom}_{U(V)}(\omega_V, C^\infty(O_Z)) = \text{Hom}_{N(X)}(\omega_V, \mathbb{C}).$$

It is easy to check that, under the identification (51), the action of E^\times on the left hand side corresponds to the following action on the right hand side:

$$a \cdot \phi := \phi \circ (\omega_V(m_a^{-1})), \quad a \in E^\times, \phi \in \text{Hom}_{N(X)}(\omega_V, \mathbb{C}).$$

By Lemma 5.1, $\text{Hom}_{N(X)}(\omega_V, \mathbb{C})$ is spanned by $\lambda_{X \otimes W^\circ}$, and (21) implies that

$$\lambda_{X \otimes W^\circ} \circ (\omega_V(m_a^{-1})) = |a|_E^{-2\rho_1 d_D^2 r} \lambda_{X \otimes W^\circ}.$$

□

Lemma 5.11. *Every element $\lambda \in \text{Hom}_{U(V)}(\omega_V, C^\infty(\Gamma_r))$ satisfies*

$$a \cdot \lambda = |a|_E^{-2\rho_1 d_D^2 r} \lambda, \quad a \in E^\times.$$

Proof. Without loss of generality, assume that λ is $\text{GL}_{d-1}(\mathcal{O}_D)$ -finite. Then the image of λ is contained in $C^\infty(\Gamma_r)$, and the lemma follows by Lemma 5.10 and by considering the following product of restriction maps:

$$C^\infty(\Gamma_r) \hookrightarrow \prod_{Z \text{ is a subspace of } D^{d-1} \text{ of dimension } r-1} C^\infty(O_Z).$$

□

Lemma 5.12. *Let μ_{Γ_r} be as in Lemma 5.8. Then for every $a \in E^\times$, the pushing forward through*

$$(52) \quad \Gamma_r \rightarrow \Gamma_r, \quad x \mapsto xa^{-1}$$

maps μ_{Γ_r} to $|a|_E^{2d_D^2 r \rho_r} \mu_{\Gamma_r}$.

Proof. View E^\times as the center of $\text{GL}_{d-1}(D)$ via the diagonal embedding. Identify Γ_r with

$$(53) \quad (U(V) \times (E^\times \text{GL}_{d-1}(\mathcal{O}_D))) / S_r^\circ,$$

where $S_r^\circ := S_r \cap (U(V) \times (E^\times \text{GL}_{d-1}(\mathcal{O}_D)))$, and S_r is as in the proof of Lemma 5.8. Then the map (52) corresponds to the left translation by $a \in E^\times$ on (53).

It is routine to check that the modulus character on S_r° extends to the positive character

$$U(V) \times (E^\times \mathrm{GL}_{d-1}(\mathcal{O}_D)) \rightarrow \mathbb{R}_+^\times, \quad (g, ah) \mapsto |a|_E^{2d_D^2 r \rho_r}.$$

Then the current lemma will follow from the next lemma. \square

Lemma 5.13. ([Lo, Theorem 33D]) *Let G be a unimodular locally compact Hausdorff topological group, with a closed subgroup S . If there is a positive character $\chi : G \rightarrow \mathbb{R}_+^\times$ which extends the modulus character of S , then up to scalar multiplication, there is a unique nonzero Borel measure μ on G/S such that*

$$g_*\mu = \chi(g)\mu, \quad g \in G,$$

where g_* is the pushing forward of measures through the left translation on G/S by g .

5.6. Conclusion of proof.

Lemma 5.14. *If V does not split, then*

$$(54) \quad \mathrm{Hom}_{U(V)}(D_{1/2}^\zeta(V^{d-1}) \otimes \omega_V, 1_V) = 0;$$

if V splits and $d = 2r$, then every element λ of the left hand side of (54) satisfies

$$a \cdot \lambda = |a|_E^{-d_D^2 r^2} \lambda, \quad a \in E^\times.$$

Proof. Denote by $\mu_{V^{d-1}}$ a Haar measure on V^{d-1} . As representations of E^\times , we have that

$$\begin{aligned} & \mathrm{Hom}_{U(V)}(D_{1/2}^\zeta(V^{d-1}) \otimes \omega_V, 1_V) \\ &= \mathrm{Hom}_{U(V)}(C^\zeta(V^{d-1}) \otimes \omega_V, 1_V) \otimes \mathrm{Hom}_{\mathbb{C}}(\mu_{V^{d-1}}^{1/2}, \mathbb{C}) \\ &\cong \mathrm{Hom}_{U(V)}(C^\zeta(V^{d-1}) \otimes \omega_V, 1_V) \otimes |\cdot|_E^{-\frac{1}{2} d_D^2 d(d-1)} \\ (55) \quad &= \mathrm{Hom}_{U(V)}(C^\zeta(\Gamma) \otimes \omega_V, 1_V) \otimes |\cdot|_E^{-\frac{1}{2} d_D^2 d(d-1)} \quad (\text{by Lemma 5.5}). \end{aligned}$$

If V does not split, then the space (55) vanishes by Lemma 5.9. This proves the first assertion. Now assume that V splits and $d = 2r$. Then Lemma 5.9 implies that the space (55) embeds into

$$\begin{aligned} & \mathrm{Hom}_{U(V)}(C^\zeta(\Gamma_r) \otimes \omega_V, 1_V) \otimes |\cdot|_E^{-\frac{1}{2} d_D^2 d(d-1)} \\ &= \mathrm{Hom}_{U(V)}(D^\zeta(\Gamma_r) \otimes \omega_V, 1_V \otimes \mathbb{C} \mu_{\Gamma_r}) \otimes |\cdot|_E^{-\frac{1}{2} d_D^2 d(d-1)} \\ &= \mathrm{Hom}_{U(V)}(D^\zeta(\Gamma_r) \otimes \omega_V, 1_V) \otimes |\cdot|_E^{-\frac{1}{2} d_D^2 d(d-1) + 2 d_D^2 r \rho_r} \quad (\text{by Lemma 5.12}) \\ &= \mathrm{Hom}_{U(V)}(\omega_V, C^{-\infty}(\Gamma_r)) \otimes |\cdot|_E^{-\frac{1}{2} d_D^2 d(d-1) + 2 d_D^2 r \rho_r}. \end{aligned}$$

Therefore by Lemma 5.11, every λ in the left hand side of (54) satisfies

$$a \cdot \lambda = |a|_E^{-2\rho_1 d_D^2 r - \frac{1}{2} d_D^2 d(d-1) + 2 d_D^2 r \rho_r} \lambda = |a|_E^{-d_D^2 r^2} \lambda, \quad a \in E^\times.$$

\square

For every ϵ -Hermitian right D -vector space $(V', \langle, \rangle_{V'})$, define an ϵ -symmetric F -bilinear form on it:

$$\langle u, v \rangle_{V'_F} := \frac{\langle u, v \rangle_{V'} + \langle u, v \rangle_{V'}^\iota}{2}.$$

Define the Fourier transform $\mathcal{F}_{V'}$ on $D_{1/2}^\epsilon(V')$ by

$$\left(\frac{\mathcal{F}_{V'}(f \mu_{V'}^{\frac{1}{2}})}{\mu_{V'}^{\frac{1}{2}}} \right) (x) := \int_{V'} f(y) \psi(\langle x, y \rangle_{V'_F}) d\mu_{V'}(y), \quad f \in C^\infty(V'), x \in V',$$

where $\mu_{V'}$ is the Haar measure on V' so that $\mathcal{F}_{V'}$ preserves the following inner product on $D_{1/2}^\epsilon(V')$:

$$\langle f \mu_{V'}^{\frac{1}{2}}, f' \mu_{V'}^{\frac{1}{2}} \rangle := \int_{V'} f(x) \overline{f'(x)} d\mu_{V'}(x).$$

It is easy to check that

$$(56) \quad \mathcal{F}_{V'}(g_* \eta) = g_*(\mathcal{F}_{V'}(\eta)), \quad g \in U(V'), \eta \in D_{1/2}^\epsilon(V'),$$

and

$$(57) \quad \mathcal{F}_{V'}((a^{-1})_* \eta) = (a^\iota)_*(\mathcal{F}_{V'}(\eta)), \quad a \in E^\times, \eta \in D_{1/2}^\epsilon(V'),$$

where $g_* : D_{1/2}^\epsilon(V') \rightarrow D_{1/2}^\epsilon(V')$ is the push forward of half densities through $g : V' \rightarrow V'$, and for every $a \in E^\times$, $a_* : D_{1/2}^\epsilon(V') \rightarrow D_{1/2}^\epsilon(V')$ is the push forward of half densities through the right multiplication on V' by a .

We are now ready to prove the vanishing of the space

$$(58) \quad \text{Hom}_{U(V)}(D_{1/2}^\epsilon(V^{d-1}) \otimes \omega_V, 1_V).$$

By Lemma 5.14, we may assume that V splits. Let λ be an element of (58). Lemma 5.14 then implies that

$$(59) \quad a \cdot \lambda = |a|_E^{-d_B^2 r^2} \lambda, \quad a \in E^\times.$$

Put

$$\lambda' := \lambda \circ (\mathcal{F}_{V^{d-1}} \otimes 1_{\omega_V}).$$

It is again an element of (58) by (56). Again by Lemma 5.14, we have that

$$a \cdot \lambda' = |a|_E^{-d_B^2 r^2} \lambda', \quad a \in E^\times.$$

By (57), the equation (59) implies that

$$a \cdot \lambda' = |a|_E^{d_B^2 r^2} \lambda', \quad a \in E^\times.$$

Therefore $\lambda' = 0$ and so $\lambda = 0$.

6. CONSERVATION RELATIONS

6.1. Persistence and non-vanishing in stable range. Now F is any local field of characteristic zero. We recall again Kudla's persistence principal [Ku2, Propositions 4.1 and 4.5] and non-vanishing of theta liftings in stable range ([Ku2, Propositions 4.3 and 4.5], and [PP, Theorem 1] for the archimedean case). We shall provide a short proof in the notation of this article.

Proposition 6.1. *Assume $\pi \in \text{Irr}(U'(V))$ is genuine with respect to $\mathbf{t} \in \mathfrak{W}_V$. If $\sigma, \sigma' \in \mathbf{t}$ and $\dim \sigma' \geq \dim \sigma$, then*

$$\Theta_\sigma(\pi) \neq 0 \Rightarrow \Theta_{\sigma'}(\pi) \neq 0.$$

Proof. Without loss of generality, assume that $\sigma' = \sigma + \sigma_2$, where σ_2 is the split element of dimension 2. Write $\omega, \omega', \omega_2$ for the representation in $\sigma, \sigma', \sigma_2$, respectively. Then $\omega' = \omega \hat{\otimes} \omega_2$. By definition, there is a nonzero (continuous in the archimedean case) $U'(V)$ -invariant functional on ω_2 . Therefore

$$\text{Hom}_{U'(V)}(\omega, \pi) \neq 0 \Rightarrow \text{Hom}_{U'(V)}(\omega', \pi) \neq 0.$$

□

Proposition 6.2. *Assume $\pi \in \text{Irr}(U'(V))$ is genuine with respect $\sigma \in \Omega_V$ and $\text{rank } \sigma \geq \dim_{\mathbb{D}} V$, then*

$$\Theta_\sigma(\pi) \neq 0.$$

Proof. By Lemma 2.6 and without loss of generality, we may assume that $\sigma \in \Omega_{V, \kappa}$ for some $\kappa \in \hat{K}'$. Then $\sigma = r_V^{\mathbb{V}}(\sigma_{\mathbb{V}})$ for some $\sigma_{\mathbb{V}} \in \Omega_{\mathbb{V}, \kappa}$. Now the lemma follows directly from Lemmas 4.1, 4.3 and 4.4. □

6.2. A lower bound. Theorems A, B and C are trivial when $V = 0$. Unless otherwise specified, from now on, assume that $V \neq 0$, and $U(V)$ is not a complex symplectic group or a real quaternionic symplectic group. Both of our key technical results (Propositions 4.2 and 5.1) exclude these two cases.

Recall that $d := \dim_{\mathbb{D}} V$.

Proposition 6.3. *Let $\mathbf{t}, \mathbf{t}' \in \mathfrak{W}_V$ be two different elements so that $\mathbf{t} - \mathbf{t}' \in \mathcal{K}_V$, and $\pi \in \text{Irr}(U'(V))$ be genuine with respect to \mathbf{t} (and hence genuine with respect to \mathbf{t}'). If F is non-archimedean or $U(V)$ is an orthogonal group, then*

$$n_{\mathbf{t}}(\pi) + n_{-\mathbf{t}'}(\pi^{\vee}) \geq 4\rho_d + 2.$$

If $\mathcal{K}_V \cong \mathbb{Z}$ (that is, $U(V)$ is a real symplectic group, a real unitary group, or a real quaternionic orthogonal group), then

$$n_{\mathbf{t}}(\pi) + n_{-\mathbf{t}'}(\pi^{\vee}) \geq 4\rho_d + 2 + 4\rho_1(|\mathbf{t} - \mathbf{t}'| - 1).$$

Proof. Denote by $\sigma \in \mathfrak{t}$ the enhanced oscillator representation of dimension $n_{\mathfrak{t}}(\pi)$ so that $\Theta_{\sigma}(\pi) \neq 0$. Likewise, denote by $\sigma' \in -\mathfrak{t}'$ the enhanced oscillator representation of dimension $n_{-\mathfrak{t}'}(\pi^{\vee})$ so that $\Theta_{\sigma'}(\pi^{\vee}) \neq 0$. Then it is clear that

$$\Theta_{\sigma+\sigma'}(1_V) \neq 0.$$

Therefore the result follows from Proposition 5.1. \square

6.3. An upper bound.

Lemma 6.4. *For every $\kappa \in \widehat{K}'$ and every $\pi \in \text{Irr}(U'(V))$ which is genuine with respect to some (and hence all) elements of $\mathfrak{W}_{V,\kappa}$, we have that*

$$(60) \quad n_{\kappa}(\pi) := \min\{n_{\mathfrak{t}}(\pi) \mid \mathfrak{t} \in \mathfrak{W}_{V,\kappa}\} \leq 2\rho_d + 1.$$

Proof. Assume that (60) does not hold, then

$$(61) \quad 2\rho_d \leq n_0 := \begin{cases} n_{\kappa}(\pi) - 1, & \text{D is quaternion;} \\ n_{\kappa}(\pi) - 2, & \text{otherwise.} \end{cases}$$

Pick a Witt tower $\mathfrak{t}_0 \in \mathfrak{W}_{V,\kappa}$ such that

$$n_{\mathfrak{t}_0}(\pi) = n_{\kappa}(\pi).$$

Denote by $\tilde{\mathfrak{t}}_0$ the element of $\mathfrak{W}_{V,\kappa}$ so that $r_V^{\vee}(\tilde{\mathfrak{t}}_0) = \mathfrak{t}_0$. First assume that D is not quaternion. By Part (i) of Proposition 4.2, as $U'(\mathbb{V})$ -representations,

$$(62) \quad \frac{I_{\kappa}(\frac{n_0}{2} - \rho_d)}{\sum_{\mathfrak{t}' \in \mathfrak{W}_{V,\kappa}, \mathfrak{t}' \neq \tilde{\mathfrak{t}}_0; \sigma' \in \mathfrak{t}', \dim \sigma' = n_0} R_{\sigma'}} \\ \cong \begin{cases} R_{\sigma_0}, & \text{if there exists an element } \sigma_0 \in \tilde{\mathfrak{t}}_0 \text{ of dimension } 4\rho_d - n_0; \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 4.3, we may pick a nonzero element μ in

$$\text{Hom}_{U'(V)}(I_{\kappa}(\frac{n_0}{2} - \rho_d), \pi).$$

Since $n_0 < n_{\kappa}(\pi)$, μ vanishes on the denominator of (62) by Lemma 4.4. Therefore it descends to a nonzero element of

$$\text{Hom}_{U'(V)}(R_{\sigma_0}, \pi).$$

Lemma 4.4 then implies that

$$n_{\mathfrak{t}_0}(\pi) \leq \dim \sigma_0 = 4\rho_d - n_0.$$

Since $n_0 < n_{\mathfrak{t}_0}(\pi)$, we have $n_0 < 4\rho_d - n_0$, which contradicts (61). For the rest of cases, we use Part (ii) of Proposition 4.2, and we reach the same contradiction. \square

Proposition 6.5. *For every \mathcal{K}_V coset \mathcal{T} in \mathfrak{W}_V and every $\pi \in \text{Irr}(U'(V))$ which is genuine with respect to some (and hence all) elements of \mathcal{T} , there are two different Witt towers $\mathfrak{t}, \mathfrak{t}' \in \mathcal{T}$ such that*

$$n_{\mathfrak{t}}(\pi) + n_{\mathfrak{t}'}(\pi) \leq 4\rho_d + 2.$$

Proof. By Lemma 2.6 and without loss of generality, assume that $\mathcal{T} = \mathfrak{W}_{V,\kappa}$ for some $\kappa \in \widehat{K}'$. Let \mathbf{t}_0 and $\tilde{\mathbf{t}}_0$ be as in the proof of Lemma 6.4. We have from Lemma 6.4 that

$$m_0 := 4\rho_d + 2 - n_{\mathbf{t}_0}(\pi) \geq 2\rho_d + 1.$$

First assume that D is not quaternion. By Part (i) of Proposition 4.2, as $U'(\mathbb{V})$ -representations,

$$(63) \quad \frac{I_\kappa(\frac{m_0}{2} - \rho_d)}{\sum_{\mathbf{t}' \in \mathfrak{W}_{V,\kappa}, \mathbf{t}' \neq \tilde{\mathbf{t}}_0; \sigma' \in \mathbf{t}', \dim \sigma' = m_0} R_{\sigma'}} \cong \begin{cases} R_{\sigma_0}, & \text{if there exists an element } \sigma_0 \in \tilde{\mathbf{t}}_0 \text{ of dimension } 4\rho_d - m_0; \\ 0, & \text{otherwise.} \end{cases}$$

Again, we pick a nonzero element μ in

$$\text{Hom}_{U'(V)}(I_\kappa(\frac{m_0}{2} - \rho_d), \pi).$$

Suppose that the proposition does not hold for π and \mathcal{T} . Then by Lemma 4.4, μ vanishes on the denominator of (63), and so descends to a nonzero element of

$$\text{Hom}_{U'(V)}(R_{\sigma_0}, \pi).$$

This is not possible as

$$\dim \sigma_0 = 4\rho_d - m_0 = n_{\mathbf{t}_0}(\pi) - 2.$$

Now assume that $U(V)$ is a non-archimedean quaternionic group or a real quaternionic orthogonal group. By Part (ii) of Proposition 4.2, as $U'(\mathbb{V})$ -representations,

$$(64) \quad \frac{I_\kappa(\frac{m_0}{2} - \rho_d)}{\sum_{\mathbf{t}' \in \mathfrak{W}_{V,\kappa}; \sigma' \in \mathbf{t}', \dim \sigma' = m_0} R_{\sigma'}} \cong \bigoplus_{\mathbf{t} \in \mathfrak{W}_{V,\kappa}; \sigma \in \mathbf{t}, \dim \sigma = 4\rho_d - m_0} R_\sigma.$$

We observe that m_0 has a different parity with $n_{\mathbf{t}_0}(\pi)$ (since $4\rho_d$ is odd for D quaternion), and so for all the contributing terms in the denominator of (64), we have $\mathbf{t}' \neq \tilde{\mathbf{t}}_0$. We may use the same argument (as in the first case) and conclude that the current proposition holds in this second case. \square

6.4. Proof of main theorems. Assume that we are in the settings of Theorems A and B and so $\mathcal{K}_V = \{1, \mathbf{t}_V\}$. Let $\mathbf{t}_1, \mathbf{t}_2$ be two elements of \mathfrak{W}_V with difference \mathbf{t}_V . They form a \mathcal{K}_V -coset. Assume that $\pi \in \text{Irr}(U'(V))$ is genuine with respect to \mathbf{t}_1 (and hence to \mathbf{t}_2). By Propositions 6.3 and 6.5, we have

$$(65) \quad \begin{cases} n_{\mathbf{t}_1}(\pi) + n_{-\mathbf{t}_2}(\pi^\vee) \geq 4\rho_d + 2, \\ n_{\mathbf{t}_2}(\pi) + n_{-\mathbf{t}_1}(\pi^\vee) \geq 4\rho_d + 2, \\ n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) \leq 4\rho_d + 2, \\ n_{-\mathbf{t}_1}(\pi^\vee) + n_{-\mathbf{t}_2}(\pi^\vee) \leq 4\rho_d + 2. \end{cases}$$

This forces all the above inequalities to be equalities, and we conclude the proof of Theorems A and B.

Lemma 6.6. *Assume that F is archimedean. For every $\pi \in \text{Irr}(U'(V))$ which is genuine with respect to $\sigma \in \Omega_V$, we have*

$$\Theta_\sigma(\pi) \neq 0 \quad \text{if and only if} \quad \Theta_{\sigma^\vee}(\pi^\vee) \neq 0.$$

Proof. This follows by using MVW-involutions on archimedean metaplectic groups and classical groups (cf. [MVW, Pr1, Su1]). MVW involutions for real quaternionic groups are discussed in [LST]. \square

We remark in passing that MVW-involutions do not exist in the non-archimedean quaternionic case. Nonetheless the assertion of Lemma 6.6 is still valid for this case, in view of the equalities in (65).

Now assume that we are in the settings of Theorem C. Thus $F = \mathbb{R}$ and $\mathcal{K}_V \cong \mathbb{Z}$, and we are given a \mathcal{K}_V -coset \mathcal{T} and $\pi \in \text{Irr}(U'(V))$ which is genuine with respect to some (and hence all) elements of \mathcal{T} .

By Proposition 6.5, there are two different Witt towers $\mathbf{t}, \mathbf{t}' \in \mathcal{T}$ such that

$$(66) \quad n_{\mathbf{t}}(\pi) + n_{\mathbf{t}'}(\pi) \leq 4\rho_d + 2.$$

By Proposition 6.3, we have

$$n_{\mathbf{t}_1}(\pi) + n_{-\mathbf{t}_2}(\pi^\vee) \geq 4\rho_d + 2 + 4\rho_1(|\mathbf{t}_1 - \mathbf{t}_2| - 1).$$

for any two different elements $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}$. Lemma 6.6 implies that

$$n_{-\mathbf{t}_2}(\pi^\vee) = n_{\mathbf{t}_2}(\pi).$$

Therefore

$$n_{\mathbf{t}_1}(\pi) + n_{\mathbf{t}_2}(\pi) \geq 4\rho_d + 2 + 4\rho_1(|\mathbf{t}_1 - \mathbf{t}_2| - 1).$$

This also forces the inequality in (66) to be an equality, and the proof of Theorem C is now complete.

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